One-Loop Calculation of Perturbative PDFs in Feynman Gauge: Unpolarized Case

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I. GAUGE-INVARIANT DEFINITIONS OF UNPOLARIZED PDFS

The gauge-invariant definitions of unpolarized PDFs are

$$f_{i/P}(x) = \int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{-ixP^+w^-} \left\langle P \left| \overline{\psi}_i(0^+, w^-, \mathbf{0}_T) \frac{\gamma^+}{2} W_F[w^-,\, 0]\, \psi_i(0) \right| P \right\rangle \tag{1a}$$

$$f_{g/P}(x) = \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{-ixP^+w^-} \left\langle P \left| G^{+j}(0^+, w^-, \mathbf{0}_T)\, W_A[w^-, 0]\, G^{+j}(0) \right| P \right\rangle \tag{1b}$$

where the subscript i means quark of flavor i, and the superscript $j = (1, 2)$ is the transverse Lorentz index of gluon field strength and is summed over. $|P\rangle$ is the proton (or any hadron/elementary *target*) state with momentum P. The spin state is not specified; in fact, the expectation values of unpolarized PDF operators do not depend on the spin of the target: one can use any pure spin state, or average (weighted or unweighted) over the spin states.

Some points to note here:

a. Light-Cone Coordinates. We are using light-cone coordinates defined using two light-cone vectors

$$n^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1),\ \bar{n}^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$$

For a general 4-vector V^μ, we define

$$V^+ = n \cdot V = \frac{V^0 + V^3}{\sqrt{2}},\ V^- = \bar{n} \cdot V = \frac{V^0 - V^3}{\sqrt{2}},\ \boldsymbol{V}_T = (V^1, V^2) \tag{2}$$

We generally write $V = (V^+, V^-, \boldsymbol{V}_T)$. So $n = (0^+, 1^-, \mathbf{0}_T)$, $\bar{n} = (1^+, 0^-, \mathbf{0}_T)$. The inner product of two Lorentz vectors V and W is

$$V \cdot W = V^+W^- + V^-W^+ - \boldsymbol{V}_T \cdot \boldsymbol{W}_T, \tag{3}$$

and in particular,

$$V^2 = 2V^+V^- - \boldsymbol{V}_T^2. \tag{4}$$

b. Dimension counting. Since we will perform the calculation in d-dimensions in dimensional regularization, it is instructive to show that PDFs are dimensionless in d-dimensions.

- *Fields.*

$$[\psi] = [\bar{\psi}] = \frac{d-1}{2} = \frac{3}{2} - \varepsilon,\ \ [A_\mu] = \frac{d-2}{2} = 1 - \varepsilon,\ [G_{\mu\nu}] = \frac{d}{2} = 2 - \varepsilon. \tag{5}$$

- *States.* The target states $|P\rangle$ are normalized by

$$\langle P'|P\rangle = (2\pi)^{d-1} 2E_P \delta^{d-1}(\boldsymbol{P} - \boldsymbol{P'}), \tag{6}$$

so

$$[\,|P\rangle\,] = [\,\langle P|\,] = \frac{1}{2}(1+1-d) = -1+\varepsilon. \tag{7}$$

- *Wilson Lines.* Since Wilson line is an exponentiatial of a dimensionless quantity, it does not carry dimension.

- *PDFs.*

$$[f_q(x)] = -1 + 2(-1+\varepsilon) + 2\left(\frac{3}{2} - \varepsilon\right) = 0; \tag{8}$$

$$[f_g(x)] = -1 - 1 + 2(-1+\varepsilon) + 2\,(2-\varepsilon) = 0. \tag{9}$$

The first -1 terms in both equations come from the integration measure of w^-, which is the light-cone coordinate, and the second -1 term in (9) is from the $1/P^+$ factor. This shows the necessity of adding $1/P^+$ factor in the definition of gluon density, while the $1/x$ is added to give $f_g(x)$ a correct probability interpretation, which will be confirmed in deriving the sum rule.

c. *Wilson Lines.*

$$\left(W_F[w^-,\,0]\right)_{jk} = P\exp\left\{-ig\int_0^{w^-} dy^- A_a^+(0^+,\,y^-,\,\mathbf{0}_T)(T_F^a)_{jk}\right\} \tag{10}$$

$$\left(W_A[w^-,\,0]\right)_{bc} = P\exp\left\{-ig\int_0^{w^-} dy^- A_a^+(0^+,\,y^-,\,\mathbf{0}_T)(T_A^a)_{bc}\right\} \tag{11}$$

where j,k are color indices in the fundamental representation and b,c are color indices in the adjoint representation:

$$j,k = 1,\cdots,N_c; \quad b,c = 1,\cdots,N_c^2-1$$

T_F and T_A are generators of the fundamental and adjoint representations, respectively, and especially[1],

$$(T_A^a)_{bc} = -if_{abc}.$$

[1] Throughout this note, we do not distinguish upper and lower indices of the color factors, i.e., $f^{abc} = f_{abc} = f^a_{bc}$, etc.

The Wilson lines collect (the component $n\cdot A$ of) gluon fields along the light cone direction n. For a fixed path, the Wilson line only depends on the two end points 0 and w^-. Under a gauge transformation, the Wilson line transforms as

$$W_R(a,b) \to W'_R(a,b) = U_R(a)\, W_R(a,b)\, U_R(b)^{-1} \tag{12}$$

where $U_R(x) = \exp\{-i\theta^a(x) T^a_R\}$ is the gauge transformation for an arbitrary representation R and $W_R(a,b)$ is the corresponding Wilson line. Note that the generators of adjoint representations are purely imaginary, so the gauge transformation $U_A(x) \equiv D(x)$ is real, and thus

$$D(x)^T = D(x)^{-1} \tag{13}$$

d. Gauge Invariance. We have left the color indices implicitly summed over in Eq. (1). The fields transform under gauge transformations as [2]

$$\psi(x)_j \to U(x)_{jk}\, \psi(x)_k, \quad \overline{\psi}(x)_j \to \overline{\psi}(x)_k [U(x)^\dagger]_{kj} \tag{14}$$

$$G^{\mu\nu}_a(x) \to D(x)_{ab} G^{\mu\nu}_b(x) \tag{15}$$

Therefore, when inserting the Wilson lines in Eq. (1), the operators transform as [3]

$$\begin{aligned}
&\overline{\psi}_j(w^-) \left(W_F[w^-,\,0]\right)_{jk}\, \psi_k(0) \\
\to&\overline{\psi}_{\bar{j}}(w^-) \left[U^\dagger(w^-)\right]_{\bar{j}j} \left\{U(w^-)_{jl} \left(W_F[w^-,\,0]\right)_{lm} \left[U(0)^\dagger\right]_{mk}\right\} [U(0)]_{k\bar{k}}\, \psi_{\bar{k}}(0) \\
=&\overline{\psi}_j(w^-) \left(W_F[w^-,\,0]\right)_{jk}\, \psi_k(0),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&G^{+j}_b(w^-)\, \left(W_A[w^-,0]\right)_{bc}\, G^{+j}_c(0) \\
\to&\left\{D_{b\bar{b}}(w^-) G^{+j}_{\bar{b}}(w^-)\right\} \left\{D_{bd}(w^-) \left(W_A[w^-,0]\right)_{de} D^T(0)_{ec}\right\} \left\{D_{c\bar{c}}(0) G^{+j}_{\bar{c}}(0)\right\} \\
=&G^{+j}_{\bar{b}}(w^-) \left\{D^T(w^-)_{\bar{b}b} D_{bd}(w^-)\right\} \left(W_A[w^-,0]\right)_{de} \left\{D^T(0)_{ec} D_{c\bar{c}}(0)\right\} G^{+j}_{\bar{c}}(0) \\
=&G^{+j}_b(w^-)\, \left(W_A[w^-,0]\right)_{bc}\, G^{+j}_c(0),
\end{aligned} \tag{17}$$

showing manifest gauge invariance. This also demenstrates that we need to sum over the color indices in order to get gauge invariant PDFs.

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[2] My convention is that $U(x) = U_F(x)$ and $D(x) = U_A(x)$ are the gauge transformations of fundamental and adjoint representations, respectively.

[3] Here we suppress the flavor index of quark field. The subscripts j, k are color indices.

e. Lorentz Invariance. Although we can see explicit Lorentz indices ($+$ and j) in Eq. (1), the definitions are in fact Lorentz invariant. To see that, we can write them in Lorentz covariant forms

$$f_{i/P}(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-ix\lambda(n\cdot P)} \left\langle P \left| \overline{\psi}_i(\lambda n) \frac{\gamma\cdot n}{2} W_F[\lambda n,\, 0]\, \psi_i(0) \right| P \right\rangle \tag{18a}$$

$$f_{g/P}(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi x (n\cdot P)} e^{-ix\lambda(n\cdot P)} \left\langle P \left| n_\mu G^{\mu\nu}(\lambda n)\, W_A[\lambda n, 0]\, G_\nu{}^\rho(0) n_\rho \right| P \right\rangle \tag{18b}$$

This makes it explicit that thus defined PDFs are scalars made up of n and P. Since they are also invariant with a scaling of $n \to \alpha n$, the PDFs do not depend on the choice of n, as long as it is a light-like vector pointing to the future direction. [4]

By the definition itself, the PDFs are Lorentz invariant. However, when deriving specific factorization formulae, n usually comes as a natural choice. For example, when P is highly boosted along $+z$ direction, $P^+ (\gg P^-, P_T)$ is the large component, and we choose $n = (0, 1, 0_T)$. When P is highly boosted along $-z$ direction, $P^- (\gg P^+, P_T)$ is the large component, and then we choose n as $\bar{n} = (1, 0, 0_T)$, which gives

$$f_{i/P}(x) = \int_{-\infty}^{\infty} \frac{dw^+}{2\pi} e^{-ixP^-w^+} \left\langle P \left| \overline{\psi}_i(w^+, 0^-, \mathbf{0}_T) \frac{\gamma^-}{2} W_F[w^+,\, 0]\, \psi_i(0) \right| P \right\rangle \tag{19a}$$

$$f_{g/P}(x) = \int_{-\infty}^{\infty} \frac{dw^+}{2\pi x P^-} e^{-ixP^-w^+} \left\langle P \left| G^{-j}(w^+, 0^-, \mathbf{0}_T)\, W_A[w^+, 0]\, G^{-j}(0) \right| P \right\rangle \tag{19b}$$

This form is useful when dealing with hadron-hadron collisions.

A. Interpretation as cut amplitude

Since the operators defining the PDFs in Eq. (1) are not time ordered, they correspond to cut amplitudes. To see this, we first write the Wilson line as

$$W_n(w^-, 0) = W_n(w^-, \infty) W_n(\infty, 0) \tag{20}$$

each one still aligning along the n direction, indicated by the subscript 'n'. Then the matrix elements are

$$\begin{aligned} &\langle P | \overline{\psi}_i(w^-) \frac{\gamma^+}{2} W_F[w^-,\, 0]\, \psi_i(0) | P \rangle \\ = &\sum_X \left\langle P \left| \overline{\psi}_i(w^-) W_F^n[w^-, \infty] \right| X \right\rangle \frac{\gamma^+}{2} \left\langle X \left| W_F^n[\infty,\, 0]\, \psi_i(0) \right| P \right\rangle \end{aligned} \tag{21}$$

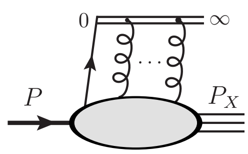
[4] Pointing n to the past direction amounts to $n \to -n$, which changes λ to $-\lambda$ everywhere. This reverses the integration direction to $\infty \to -\infty$ and introduces an extra minus sign.

$$
\begin{aligned}
&\left\langle P \left| G^{+j}(w^-)\, W_A[w^-,0]\, G^{+j}(0) \right| P \right\rangle \\
&= \sum_X \left\langle P \left| G^{+j}(w^-)\, W_A^n[w^-,\infty] \right| X \right\rangle \left\langle X \left| W_A^n[\infty,\,0]\, G^{+j}(0) \right| P \right\rangle
\end{aligned} \tag{22}
$$

$\langle X\, | W_F^n[\infty,\, 0]\, \psi_i(0) |\, P\rangle$ is the amplitude of annihilating a quark of flavor i (with ψ_i) in the proton (and possible gluons with the Wilson line $W_F^n[\infty,\, 0]$) and creating a final state $|X\rangle$. $\left\langle P \left| \overline{\psi}_i(w^-) W_F^n[w^-,\infty] \right| X \right\rangle$ corresponds to its Hermitian conjugate. So the normal ordering corresponds to an amplitude squared, with a sum over the *final states.* In terms of diagrams, we use a cut line to label on-shell final states and a sum over them.

We denote

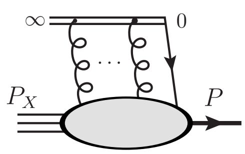
$$M_L = $$



$$= \langle X\, | W_F^n[\infty,\, 0]\, \psi_i(0) |\, P\rangle \tag{23}$$

as the amplitude of annihilating a quark from a proton and arriving at the final state X, and

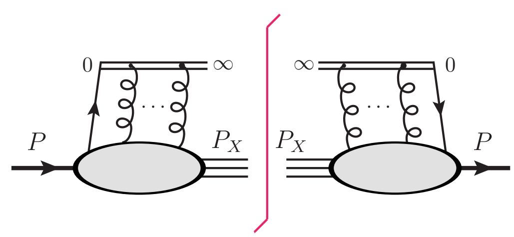
$$M_R = $$



$$= \left\langle P \left| \overline{\psi}_i(0) W_F^n[0,\infty] \right| X \right\rangle = M_L^\dagger \gamma^0 \tag{24}$$

as its Hermitian conjugate. Summing over the final states X, with a δ function $(2\pi)^4\delta^4(P - P_X - k)$ (and a parton vertex Γ) gives the probability of getting a quark of momentum k

$$\sum_X (2\pi)^4 \delta^4(P - P_X - k) M_R \Gamma M_L \equiv$$



$$
\begin{aligned}
&= \sum_X \int d^4 w\, e^{i(P-P_X-k)\cdot w} \left\langle P \left| \overline{\psi}_i(0) W_F^n[0,\infty] \right| X \right\rangle \Gamma \left\langle X \left| W_F^n[\infty,\, 0]\, \psi_i(0) \right| P \right\rangle \\
&= \sum_X \int d^4 w\, e^{-ik\cdot w} \left\langle P \left| \overline{\psi}_i(w) W_F^n[w,\infty] \right| X \right\rangle \Gamma \left\langle X \left| W_F^n[\infty,\, 0]\, \psi_i(0) \right| P \right\rangle \\
&= \int d^4 w\, e^{-ik\cdot w} \left\langle P \left| \overline{\psi}_i(w)\, \Gamma\, W_F^n[w,\, 0]\, \psi_i(0) \right| P \right\rangle
\end{aligned} \tag{25}
$$

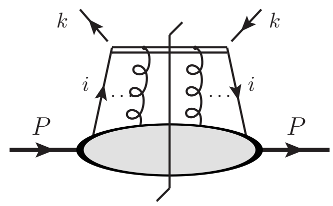
where [5] we have defined the cut as

$$\text{cut line} = \text{put on-shell and } \sum_X (2\pi)^4 \delta^4(P - P_X - k) \tag{26}$$

Then integrating over k with a δ function gives the quark parton density

$$f_{i/P}(x) = \int \frac{d^4k}{(2\pi)^4} \delta(k^+ - xP^+) \left\{ \int d^4w \, e^{-ik\cdot w} \left\langle P \left| \overline{\psi}_i(w) \, \frac{\gamma^+}{2} \, W_F^n[w,\, 0] \, \psi_i(0) \right| P \right\rangle \right\}$$

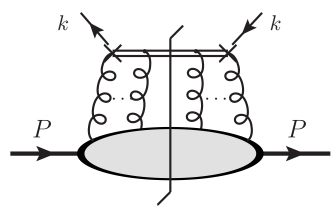
$$= \int \frac{d^4k}{(2\pi)^4} \delta(k^+ - xP^+) \tag{27}$$



The diagram is a cut diagram (in momentum space) with momentum k flowing out of the *composite* quark-link vertex to the left of the cut line and into the vertex to the right. The parton density is obtained by integrating over the momentum k, with a $\delta(k^+ - xP^+)$ factor to fix k^+. Note that the k is not necessarily the momentum carried by the parton line. Similarly, the gluon density is

$$f_{g/P}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{xP^+} \delta(k^+ - xP^+) \left\{ \int d^4w \, e^{-ik\cdot w} \left\langle P \left| G^{+j}(w) \, W_A^n[w,\, 0] \, G^{+j}(0) \right| P \right\rangle \right\}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{xP^+} \delta(k^+ - xP^+) \tag{28}$$



Note that we have added the $\gamma^+/2$ and $1/xP^+$ factors (and an implicit factor $\delta_{jj'}$ to sum over the gluon spins) into the definitions. We call them *parton vectices*. They are to project out certain parton densities from the general cut amplitude.

This understanding is the first step in calculating parton densities from their definitions. **First we calculate the cut amplitude, and then integrate over the momentum k with appropriate parton vertices and $\delta(k^+ - xP^+)$.**

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[5] One should note that we have tacitly identified those two ∞ points with each other in Eq. (25) after summing over states X in the last step. This is not completely justified, because we keep using Wilson lines along n direction, so if w has non-zero $+$ or transverse components, these two Wilson lines will not meet at infinity. However, since we are going to integrate over k^- and $\boldsymbol{k}_T$, this issue is not a big problem. This only requires careful treatment when dealing with k_T dependent parton densities, where only k^- is to be integrated over.

Note: The external parton lines are NOT amputated or on shell. We need to assign propagators to them and include high-order corrections.

B. Anti-quark PDF

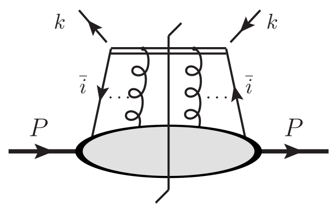
The above discussion from Eq. (23) to (27) is easy to be generalized to anti-quark PDF with the replacement

$$M_L \to \left\langle X \left|\overline{\psi}_i(0) W_F^n[0,\, \infty]\right| P\right\rangle,\ M_R \to \langle P\, | W_F^n[\infty,\, 0]\, \psi_i(0) |\, X\rangle\,. \tag{29}$$

This leads to

$$\begin{aligned} f_{\bar{i}/P}(x) &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{-ixP^+w^-} \mathrm{tr}_{color} \mathrm{Tr}_{spin} \left[\frac{\gamma^+}{2} \left\langle P \,\middle|\, W_F^n[\infty,\, w^-] \psi_i(0^+, w^-, \mathbf{0}_T) \overline{\psi}_i(0) W_F^n[0,\, \infty] \middle|\, P\right\rangle \right] \\ &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{-ixP^+w^-} \mathrm{tr}_{color} \mathrm{Tr}_{spin} \left[\frac{\gamma^+}{2} \left\langle P \,\middle|\, W_F^n[0,\, w^-] \psi_i(0^+, w^-, \mathbf{0}_T) \overline{\psi}_i(0) \middle|\, P\right\rangle \right] \end{aligned} \tag{30}$$

$$= \int \frac{d^4k}{(2\pi)^4} \delta(k^+ - xP^+) \tag{31}$$



which only differs from Eq. (27) in reversing the fermion line arrows.

Now we assume that **fields along the minus n light cone commute or anticommute.** [6] This gives that

$$\begin{aligned} f_{\bar{i}/P}(x) &= -\int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{-ixP^+w^-} \mathrm{tr} \left[\frac{\gamma^+}{2} \left\langle P \,\middle|\, \overline{\psi}_i(0) W_F^n[0,\, w^-] \psi_i(w^-) \middle|\, P\right\rangle \right] \\ &= -\int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{-ixP^+w^-} \left\langle P \,\middle|\, \overline{\psi}_i(-w^-) \frac{\gamma^+}{2} W_F^n[-w^-,\, 0] \psi_i(0) \middle|\, P\right\rangle \\ &= -\int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{+ixP^+w^-} \left\langle P \,\middle|\, \overline{\psi}_i(w^-) \frac{\gamma^+}{2} W_F^n[w^-,\, 0] \psi_i(0) \middle|\, P\right\rangle \\ &= -f_{i/P}(-x) \end{aligned} \tag{32}$$

where in the second line we used the translation invariance to translate the operator by w^-. Note that we have not only used the anticommutation relation between fermion fields, but

[6] This is actually not an assumption. There are several ways to justify it. For an example, light-cone quantization takes it as the starting point. Another way is to regard the light-cone as a limit of space-like separation, for which the fields commute or anticommute. I am not able to give a fully-justified argument within the canonical quantization formalism.

also that fermion fields commute with the gluon fields (in the Wilson line) along the n light cone.

Gluon is its own antiparticle. By using the commutation relation we also have

$$\begin{aligned} f_{g/P}(x) &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{-ixP^+w^-} \left\langle P \left| G_c^{+j}(0)\, G_b^{+j}(w^-)\, W_A[w^-,0]_{bc} \right| P \right\rangle \\ &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{-ixP^+w^-} \left\langle P \left| G_c^{+j}(0) \left(W_A[w^-,0]\right)^T{}_{cb}\, G_b^{+j}(w^-) \right| P \right\rangle \\ &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{-ixP^+w^-} \left\langle P \left| G_c^{+j}(0) \left(W_A[0,\, w^-]\right)_{cb}\, G_b^{+j}(w^-) \right| P \right\rangle \\ &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{-ixP^+w^-} \left\langle P \left| G_c^{+j}(-w^-) \left(W_A[-w^-,\, 0]\right)_{cb}\, G_b^{+j}(0) \right| P \right\rangle \\ &= \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{+ixP^+w^-} \left\langle P \left| G_c^{+j}(w^-) \left(W_A[w^-,\, 0]\right)_{cb}\, G_b^{+j}(0) \right| P \right\rangle \\ &= -\int_{-\infty}^{\infty} \frac{dw^-}{2\pi(-x) P^+} e^{-i(-x)P^+w^-} \left\langle P \left| G_c^{+j}(w^-) \left(W_A[w^-,\, 0]\right)_{cb}\, G_b^{+j}(0) \right| P \right\rangle \\ &= -f_{g/P}(-x) \end{aligned} \tag{33}$$

Note that on the third line we have used the fact that the Wilson line in the adjoint representation is real, cf. Eq. (13). We used the commutation relation between transverse gluon field components, and between transverse components and + components (in the Wilson line). To summarize, we have

$$f_{\bar{j}/P}(x) = -f_{j/P}(-x) \tag{34}$$

for $j = q, g$. This can be verified/used also in perturbative PDFs.

C. Support Property

Eq. (34) can be used to prove another property of PDF, that is,

$$f_{j/P}(x) \neq 0 \text{ only when } -1 \leq x \leq 1 \tag{35}$$

First, from Eq. (25) we see that the momentum flowing to the final state X is $P - k$. Since $|X\rangle$ is a physical on-shell state, it must have a positive P_X^+ momentum component, so

$$(P-k)^+ = (1-x)P^+ \geq 0 \tag{36}$$

and thus

$$x \leq 1 \tag{37}$$

This comes from the cut-amplitude interpretation and holds for both parton and anti-parton PDFs. It seems that there is no constraint when $x < 0$. However, when $x < 0$ it can be connected to the anti-parton PDF. From Eq. (34), we have

$$f_{j/P}(x) = -f_{\bar{j}/P}(-x) \tag{38}$$

where the right-hand side is non-zero only when $-x \leq 1$, i.e., $x \geq -1$. Therefore, we get Eq. (35).

D. Interpretation as un-cut diagram

The commutation/anticommutation relations also mean that we can add time ordering to the operators without changing the matrix elements, so Eq. (1) can also be written as

$$f_{i/P}(x) = \int_{-\infty}^{\infty} \frac{dw^-}{2\pi} e^{-ixP^+w^-} \left\langle P \left| T \left\{ \overline{\psi}_i(0^+, w^-, \mathbf{0}_T) \frac{\gamma^+}{2} W_F[w^-, 0]\, \psi_i(0) \right\} \right| P \right\rangle \tag{39a}$$

$$f_{g/P}(x) = \int_{-\infty}^{\infty} \frac{dw^-}{2\pi x P^+} e^{-ixP^+w^-} \left\langle P \left| T \left\{ G^{+j}(0^+, w^-, \mathbf{0}_T)\, W_A[w^-, 0]\, G^{+j}(0) \right\} \right| P \right\rangle \tag{39b}$$

The time-ordering means that we can connect them to uncut diagrams. That is, **we can remove the cut lines in Eq.** (27)(28)(31). This can be verified by explicit calculations. At calculation level, the equivalence between uncut diagrams and cut diagrams is due to the integration over k^-. The integral over k^- is non-zero only when k^- has poles on different sides of the real axis in complex plane. That sets x to be between -1 and 1. When the integrand does have poles on different sides, the selection of a certain pole is equivalent to cutting the corresponding propogator.

II. MULTIPLICATIVE RENORMALIZATION

The PDFs defined in Eq. (1) are bare PDFs, defined using bare fields. They contain UV divergences and require renormalization. The renormalization of PDF is separate from QCD renormalization. It goes by removing the UV divergences from the perturbative expansion. The result turns out to be multiplicative

$$f_{j/P}(x) = \sum_{j'} \int_x^1 \frac{dz}{z} Z_{jj'}\left(\frac{x}{z}\right) f^{(0)}_{j'/P}(z) \tag{40}$$

Here we use explicit superscript '(0)' to indicate that $f^{(0)}_{j'/P}(z)$ is the bare PDF defined with bare fields. $Z_{jj'}(z)$ is a convolution factor that removes the UV divergences in $f^{(0)}_{j'/P}(z)$. The convolution form suggests that the UV divergences are functions of x. In $\overline{\text{MS}}$ scheme, we will only remove divergence proportional to

$$\frac{1}{\varepsilon} - \gamma + \ln 4\pi$$

Following Collins' book [1], we write this factor as

$$\frac{S_\varepsilon}{\varepsilon} \equiv \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \frac{1}{\varepsilon} = \frac{1}{\varepsilon} - \gamma + \ln 4\pi + \mathcal{O}(\varepsilon) \tag{41}$$

It is convenient to do the calculation using renormalized fields, for which the Feynman rules are with renormalized couplings plus counterterm interactions. So we rewrite Eq. (40) as

$$f_{j/P}(x) = \sum_{j'} \int_x^1 \frac{dz}{z} Z_{jj'}\left(\frac{x}{z}\right) Z_{2j'} \left(Z^{-1}_{2j'}\, f^{(0)}_{j'/P}(z)\right) \tag{42}$$

where $Z_{2j'}$ is the field renormalization factor of field j', and $\left(Z^{-1}_{2j'} f^{(0)}_{j'/P}(z)\right)$ stands for PDFs defined with renormalized fields (we do not introduce extra symbols to avoid confusion). *Later in the specific calculations, we write* $f^{(0)}_{j/P}(z)$ *to mean* $\left(Z^{-1}_{2j} f^{(0)}_{j/P}(z)\right)$, *also to avoid cumbersome notations.*

Since $Z_{jj'}(z)$ does not depend on the target property, we are allowed to use on-shell parton state as an elementary target. For that we have

$$f_{j/k}(x) = \sum_{j'} \int_x^1 \frac{dz}{z} Z_{jj'}\left(\frac{x}{z}\right) Z_{2j'} \left(Z^{-1}_{2j'}\, f^{(0)}_{j'/k}(z)\right) \tag{43}$$

At leading order (LO), there is no divergence, and we have (see Eq. (128) below)

$$f^{[0]}_{j/k}(x) = f^{[0]}_{(0)j/k}(x) = \left(Z^{-1}_{2j}\, f^{(0)}_{j/k}\right)^{[0]}(x) = \delta_{jk}\delta(1-x) \tag{44}$$

where the superscript '$[n]$' means at n-th order of α_s. This gives

$$Z^{[0]}_{jj'}(z) = \delta_{jj'}\delta(1-z) \tag{45}$$

At next-to-leading order (NLO), we have

$$
\begin{aligned}
f^{[1]}_{j/k}(x) &= \sum_{j'} \int_x^1 \frac{dz}{z} \left\{ \left[Z_{jj'}\left(\frac{x}{z}\right) Z_{2j'} \right]^{[1]} \left(Z^{-1}_{2j'}\, f^{(0)}_{j'/k} \right)^{[0]} (z) + \left[Z_{jj'}\left(\frac{x}{z}\right) Z_{2j'} \right]^{[0]} \left(Z^{-1}_{2j'}\, f^{(0)}_{j'/k} \right)^{[1]} (z) \right\} \\
&= \sum_{j'} \int_x^1 \frac{dz}{z} \left\{ \left[Z_{jj'}\left(\frac{x}{z}\right) Z_{2j'} \right]^{[1]} \delta_{j'k}\delta(1-z) + \delta_{jj'}\delta(1-x/z) \left(Z^{-1}_{2j'}\, f^{(0)}_{j'/k} \right)^{[1]} (z) \right\} \\
&= \boxed{\left[Z_{jk}(x)\, Z_{2k} \right]^{[1]} + \left(Z^{-1}_{2j}\, f^{(0)}_{j/k} \right)^{[1]} (x)} \\
&= \left[Z_{jk}(x) \right]^{[1]} Z^{[0]}_{2k} + \left[Z_{jk}(x) \right]^{[0]} Z^{[1]}_{2k} + \left(Z^{-1}_{2j}\, f^{(0)}_{j/k} \right)^{[1]} (x) \\
&= \left(Z^{-1}_{2j}\, f^{(0)}_{j/k} \right)^{[1]} (x) + \left[Z_{jk}(x) \right]^{[1]} + \delta_{jk}\delta(1-x) Z^{[1]}_{2k}
\end{aligned}
\tag{46}
$$

The determination of $[Z_{jk}(x)]^{[1]}$ is through the third line, where we use $[Z_{jk}(x)\, Z_{2k}]^{[1]}$ to remove the UV divergence in $\left(Z^{-1}_{2j}\, f^{(0)}_{j/k} \right)^{[1]} (x)$, and then by substracting $Z^{[1]}_{2k}$ we can get $[Z_{jk}(x)]^{[1]}$.

A. Evolution of PDFs

The renormalization of bare PDFs introduce a renormalization scale μ, and therefore $f_{j/P}$ and $Z_{jj'}$ depend on μ. Since the bare PDFs defined in terms of bare fields do not depend on μ, we have the renormalization group equation for renormalized PDFs

$$
\mu^2 \frac{d}{d\mu^2} f_{j/P}(x) = \sum_{j'} \int_x^1 \frac{dz}{z} P_{jj'}\left(\frac{x}{z}\right) f_{j'/P}(z)
\tag{47}
$$

where $P_{jj'}$ comes through $Z_{jj'}$ by

$$
\mu^2 \frac{d}{d\mu^2} Z_{jk}(z) = \sum_{j'} \int_z^1 \frac{dz'}{z'} P_{jj'}\left(\frac{z}{z'}\right) Z_{j'k}(z')
\tag{48}
$$

At LO, by Eq. (45), we have

$$
P^{[0]}_{jj'}(z) = 0
\tag{49}
$$

At NLO Eq. (48) gives

$$
\mu^2 \frac{d}{d\mu^2} Z^{[1]}_{jk}(z) = \sum_{j'} \int_z^1 \frac{dz'}{z'} \left\{ P^{[1]}_{jj'}\left(\frac{z}{z'}\right) Z^{[0]}_{j'k}(z') + P^{[0]}_{jj'}\left(\frac{z}{z'}\right) Z^{[1]}_{j'k}(z') \right\} = P^{[1]}_{jk}(z)
\tag{50}
$$

We will see that at NLO, $Z^{[1]}_{jk}(z)$ depends on μ through $\alpha_s(\mu)$

$$
\frac{dZ^{[1]}_{jk}(z)}{d\ln\mu^2} = \frac{d\alpha_s}{d\ln\mu^2} \frac{dZ^{[1]}_{jk}(z)}{d\alpha_s} = (-\varepsilon\, \alpha_s) \frac{Z^{[1]}_{jk}(z)}{\alpha_s} = -\varepsilon\, Z^{[1]}_{jk}(z)
\tag{51}
$$

where we only kept the relevant term at order $\mathcal{O}(\alpha_s)$. When taking $\varepsilon \to 0$, this is equal to $P_{jk}^{[1]}(z)$, and hence we have (at $\overline{\text{MS}}$ scheme)

$$Z_{jk}^{[1]}(z) = -\frac{S_\varepsilon}{\varepsilon} P_{jk}^{[1]}(z) \tag{52}$$

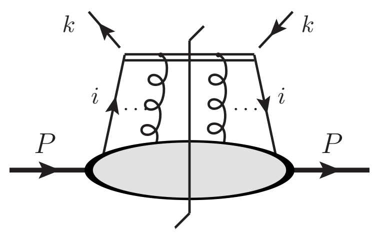
III. FEYNMAN RULES

Apart from the Feynman rules for the usual propagators and QCD vertices, there are three more elements involved in the PDFs: 1) parton vertices, 2) gluon vertex arising from the field strength, and 3) Wilson lines. Finally, we will also discuss the rules concerning the cut line.

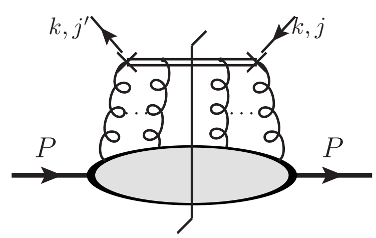
1. *Parton Vertices*

The discussion of this has been covered in Sec. I. We list the results here

$$f_{i/P}(x) = \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \delta(k^+ - xP^+) \mathrm{Tr}\, \frac{\gamma^+}{2} \tag{53a}$$



$$f_{g/P}(x) = \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \delta(k^+ - xP^+) \frac{1}{xP^+} \delta_{jj'} \tag{53b}$$



We call the factors in red fonts parton vertices.

2. *Gluon vertex arising from the field strength*

Due to gauge invariance, the external parton point (the "point" of the parton fields, in the same context as "2-point Green function") is not a single point, but a composite vertex of the parton field and gauge link. For the quark case, since we use quark field ψ or $\overline{\psi}$ to annihilate or create a quark parton, there is no special vertex when the quark parton line

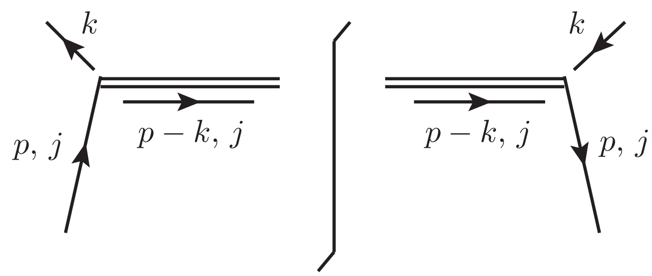


FIG. 1: Quark-link vertex = 1. Momentum and color conserve here. Momentum k flows out to the left of the cut and flows in to the right of the cut. j is the color index in the fundamental representation.

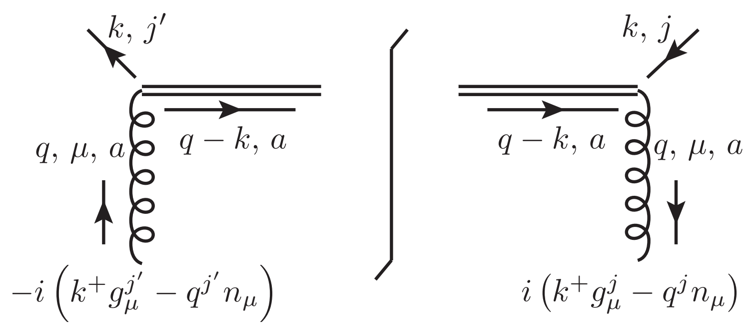


FIG. 2: Gluon-link vertex. Momentum and color conserve here. Momentum k flows out to the left of the cut and flows in to the right of the cut. j, j' are the transverse Lorentz indices that are extracted out at the vertices. a is the color index in the adjoint representation.

meets the gauge link. One just needs to remember that the color and momentum flowing through this vertex is conserved. This rule is summarized in Fig. 1.

For the vertex connecting gluon parton line and the gauge link, it is different, because we use $G^{+j}(x)$ instead of $A^j(x)$ to create or annihilate gluons. To the left of the cut line, since

$$G_a^{+j}(x) = \partial^+ A_a^j(x) - \partial^j A_a^+(x) - g f_{abc} A_b^+(x) A_c^j(x) \tag{54}$$

at the vertex where a gluon with momentum q, polarization index μ and color a is annihilated, the first two terms give

$$-iq^+ g_\mu^j - (-iq^j) g_\mu^+ = -i\left(q^+ g_\mu^j - q^j n_\mu\right)$$

Color index a flows through the gauge link smoothly. The third term in $G_a^{+j}(x)$ means that we can also have two gluon lines connecting to the gauge link at the same point, one carrying longitudinal polarization $+$ and the other carrying transverse component j. This may sound complicated, but is not new, because the gluon lines attached to the gauge link *can* meet the gluon parton line at the edge of the gauge link, and they all carry longitudinal polarization $+$. In fact, we can assign the third term into the gauge link by using the relations

$$\partial_x^+ \left(W_A^n[x,y]\right)_{ab} = -g\, f_{cad} A_c^+ \left(W_A^n[x,y]\right)_{db} \tag{55}$$

$$\partial_x^+ \left(A_a^j(x) W_A^n[x,y]_{ab}\right) = \left(\partial^+ A_a^j - g\, f_{ade}\, A_d^+(x) A_e^j(x)\right) W_A^n[x,y]_{ab} \tag{56}$$

Thus the first and third terms in $G_a^{+j}(x)$ can be combined with the Wilson line into Eq. (56). This whole term annihilates a gluon with momentum q, polarization index μ and color a, and possibly other longitudinal gluons which we assign to the Wilson line, at the vertex $x = 0$, and in total annihilates momentum k at this vertex ($q - k$ flows to the gauge link). Therefore, the vertex is

$$-ik^+ g_\mu^j - (-iq^j) g_\mu^+ = -i\left(k^+ g_\mu^j - q^j n_\mu\right) \tag{57}$$

where the first term comes from the whole term in Eq. (56), and the second term comes from the second term in Eq. (54). The vertex to the right of the cut line can be obtained by taking a Hermitian conjugate. These are summarized in Fig. 2.

3. Wilson line

We use a double line to denote the Wilson line which collects collinear longitudinal gluons. This notation makes it look like a new kind of propogator, but in fact it is not. Propogators arise from the contraction of two fields and there are momenta flowing along them, but Wilson line is just a line where (an infinite number of) gluon fields are situated. That said, we will find propagator-like Feynman rules for the Wilson line and it can be interpreted as momenta flowing along the Wilson line. So we equivalently regard Wilson line as a special kind of propogator and assign momentum to it.

A certain diagram corresponds to a certain way of Wick contraction of time-ordered fields. [7] For a diagram with n gluons attached to the Wilson line (to the left of the cut line), it

[7] Since on each side of the cut line, we only have one coordinate, we can equivalently add a time-ordering operator so that the diagram on each side of the cut line is the usual Feynman diagram. Note that the path ordering in the Wilson line commutes with time-ordering operator.

comes from the term in the expansion of W that contains n gluon fields [8]

$$W[\infty,0] \supset \frac{1}{n!}\mathcal{P}\prod_{i=1}^{n}\left\{-ign_{\mu_i}\int_0^\infty d\lambda_i\, A_{a_i}^{\mu_i}(\lambda_i n)t_{a_i}\right\}$$
$$= (-ign_{\mu_n}t_{a_n})\cdots(-ign_{\mu_1}t_{a_1})\int_0^\infty d\lambda_1\int_{\lambda_1}^\infty d\lambda_2\cdots\int_{\lambda_{n-1}}^\infty d\lambda_n A_{a_n}^{\mu_n}(\lambda_n n)\cdots A_{a_1}^{\mu_1}(\lambda_1 n) \tag{58}$$

where the path ordering gives $n!$ identical terms that cancel the $1/n!$ factor. The factor $(-ign_{\mu_i}t_{a_i})$ becomes the Feynman rule for the vertex where gluon i attaches to the gauge link. Each A field creates a corresponding gluon, in the order that gluon lines attach to the gauge link, following the path ordering rule. Suppose $A_{a_j}^{\mu_j}(\lambda_j n)$ creates a gluon with momentum k_j. (This k_j flows out of the Wilson line vertex and is treated as a loop momentum.) Then apart from the gluon propagator associated with this gluon line, we pick up a factor

$$e^{i\lambda_j n\cdot k_j}.$$

So besides the usual Feynman rules of QCD couplings and propagators, Wilson line gives extra vertices

$$-ign_{\mu_j}t_{a_j}$$

and the factor

$$F = \int_0^\infty d\lambda_1\int_{\lambda_1}^\infty d\lambda_2\cdots\int_{\lambda_{n-1}}^\infty d\lambda_n\prod_{j=1}^{n}e^{+i\lambda_j n\cdot k_j} \tag{59}$$

Using

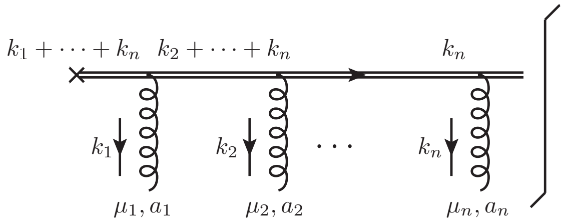
$$\int_z^\infty e^{iky} = \frac{i}{k+i\varepsilon}e^{ikz} \tag{60}$$

we have

$$F = \int_0^\infty d\lambda_1\int_{\lambda_1}^\infty d\lambda_2\cdots\int_{\lambda_{n-2}}^\infty d\lambda_{n-1}\left(\prod_{j=1}^{n-1}e^{+i\lambda_j n\cdot k_j}\right)\int_{\lambda_{n-1}}^\infty d\lambda_n e^{+i\lambda_n n\cdot k_n}$$
$$= \int_0^\infty d\lambda_1\int_{\lambda_1}^\infty d\lambda_2\cdots\int_{\lambda_{n-2}}^\infty d\lambda_{n-1}\left(\prod_{j=1}^{n-1}e^{+i\lambda_j n\cdot k_j}\right)\frac{i}{n\cdot k_n+i\varepsilon}e^{i\lambda_{n-1}n\cdot k_n}$$
$$= \frac{i}{n\cdot k_n+i\varepsilon}\cdot\frac{i}{n\cdot(k_n+k_{n-1})+i\varepsilon}\cdots\frac{i}{n\cdot(k_n+\cdots+k_1)+i\varepsilon} \tag{61}$$

[8] The g and A here are bare coupling and bare gluon field.

In terms of diagrams, the Feynman rule for Wilson line to the left of the cut is



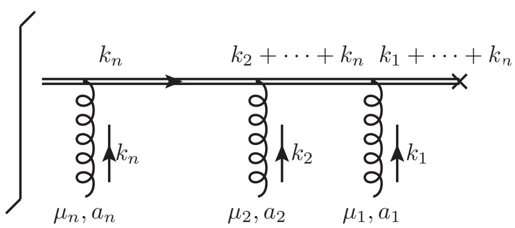
$$
\begin{aligned}
=&(-ign_{\mu_n}t_{a_n})\frac{i}{n\cdot k_n+i\varepsilon}(-ign_{\mu_{n-1}}t_{a_{n-1}})\frac{i}{n\cdot(k_n+k_{n-1})+i\varepsilon}\\
&\times\cdots\times(-ign_{\mu_1}t_{a_1})\frac{i}{n\cdot(k_n+\cdots+k_1)+i\varepsilon}
\end{aligned}
\tag{62}
$$

where we have assigned momenta to each segment of the Wilson line. One link segment with momentum k gives exactly a "propagator"

$$
\frac{i}{n\cdot k+i\varepsilon}
\tag{63}
$$

with n being the direction of the Wilson line pointing to infinity.

Similarly, we have the rule for the Wilson line to the right of the cut line, being just the Hermitian conjugate



$$
\begin{aligned}
=&(ign_{\mu_1}t_{a_1})\frac{-i}{n\cdot(k_n+\cdots+k_1)-i\varepsilon}(ign_{\mu_2}t_{a_2})\frac{-i}{n\cdot(k_n+\cdots+k_2)-i\varepsilon}\\
&\times\cdots\times(ign_{\mu_n}t_{a_n})\frac{-i}{n\cdot k_n-i\varepsilon}
\end{aligned}
\tag{64}
$$

The above derivation is true for any representation. For quark PDF, it is the fundamental representation and $t_a=((t_a)_{jk})$. For gluon PDF, it is adjoint representation and $t_a=(-if_{abc})$. From our discussion, it is clear that there is no momentum flowing to or from ∞ through the Wilson line. In other words, in a cut diagram, no momentum flows across the cut line through the gauge link.

The Feynman rules are collected in Fig. 3-5. It is important that n *appears both in the propagators and the vertices, as is necessary to guarantee Ward identity.*

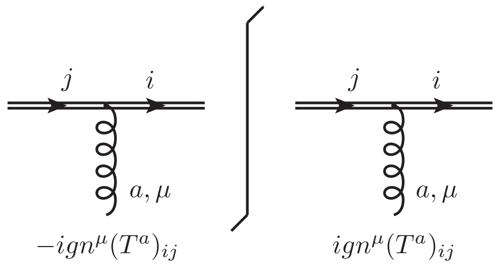


FIG. 3: The Wilson line vertex for quark PDF. T^a ($a = 1, \cdots, N_c^2 - 1$) are the generators of the fundamental representation. $i, j = 1, \cdots, N_c$ are color indices in the fundamental representation. By putting a cut line we do not mean that the two parts are Hermitian conjugate to each other, but only that the rules follow according to their positions with respect to the cut line.

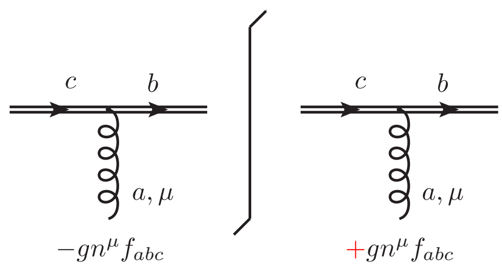


FIG. 4: The Wilson line vertex for gluon PDF. f_{abc} ($a, b, c = 1, \cdots, N_c^2 - 1$) is the structure constant of the color group. It comes by replacing T^a in Fig. 3 by $-if_{abc}$, corresponding to the adjoint representation. Note the '+' sign to the right of the cut line. By putting a cut line we do not mean that the two parts are Hermitian conjugate to each other, but only that the rules follow according to their positions with respect to the cut line.

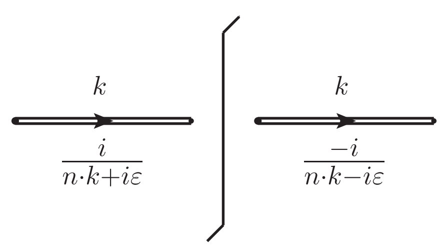
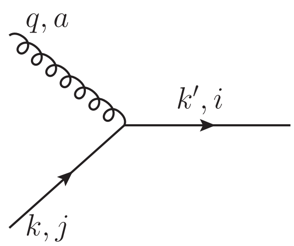


FIG. 5: The Wilson line propagator. No momentum flows across the cut line throught the Wilson line.

4. Cut rules

a. Feynman rules for the diagram on the right of the cut line. The amplitude on the right of the cut line should be understood as the complex conjugate of the part on the left. For a basic process like $q_i(k) + g_a(q) \to q_j(k')$, the amplitude is

$\mathcal{M}_{qL} =$



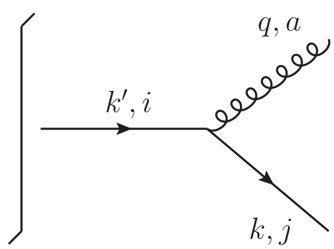
$$= \bar{u}(k') \left(-ig\mu^{\varepsilon}\gamma^{\mu}T^a_{ij}\right) u(k)\varepsilon_{\mu}(q) \tag{65}$$

where a, i, j are color indices. Taking a Hermitian/complex conjugate gives

$$\mathcal{M}^*_{qL} = \bar{u}(k) \left(ig\mu^{\varepsilon}\gamma^{\mu}T^{a*}_{ij}\right) u(k')\varepsilon^*_{\mu}(q) = \bar{u}(k) \left(ig\mu^{\varepsilon}\gamma^{\mu}T^{a}_{ji}\right) u(k')\varepsilon^*_{\mu}(q) \tag{66}$$

which can be naturally understood as its the inverse process $q_j(k') \to q_i(k) + g_a(q)$, with i changed to $-i$ in the vertex, i.e.,

$\mathcal{M}^*_{qL} =$



$$\equiv \mathcal{M}_{qR} \tag{67}$$

where we add a cut line to indicate the change of Feynman rules for the amplitude to its right. Note that this simple rule relies on T^a being a Hermitian matrix, and that there is a γ^0 factor defined in $\bar{u}$, which keeps γ^{μ} as γ^{μ}. When there is a chain of γ matrices like

$$\bar{u}(k')\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\gamma_n}u(k) \tag{68}$$

the complex conjugate simply gives

$$\bar{u}(k)\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\gamma_1}u(k) \tag{69}$$

corresponding perfectly to the inverse process. So as a first (not complete) rule, we remember that for the vertices to the right of the cut we replace all i to $-i$. When there involve propagators, it is also true, because propagators only add some γ matrices and c-number coefficients. When taking a complex conjugate, we simply reverse the process and replace all i to $-i$.

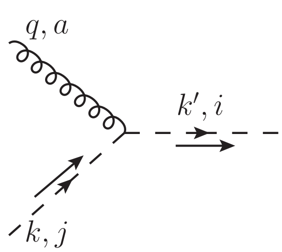
We emphasize the importance of Hermiticity of T^a_{ij}. Taking a complex conjugate gives

$$T^{a*}_{ij} = (T^{a\dagger})_{ji} = T^a_{ji} \tag{70}$$

so keeping the form of T^a naturally reverse the initial and final states. Confusion can arise when we deal with the adjoint representation, for which the generator $(T^a_A)_{bc} = -if^{abc}$ where f^{abc} is real. The $-i$ cancels the i in the vertex and then there is no factor of i in the vertex. But note that when writing down $(T^a_A)_{bc}$, b is the color index of the final state particle and c is of initial state particle, [9] while when reversing the process to the right of the cut line, b, c ordering becomes opposite. Then when writing down the amplitude we should have $(T^a_A)_{cb}$, which finally gives $-f^{abc}$. This effect appears in the coupling of gluon to the gauge link, and gluon-ghost vertex.

Another complication arises when the vertex depends on momenta, like scalar QED/QCD vertex and tri-gluon vertex. Still taking the simple process $\widetilde{q}_i(k) + g_a(q) \to \widetilde{q}_j(k')$, in which $\widetilde{q}$ stands for a scalar colored particle, we have the amplitude

$\mathcal{M}_{sL} =$



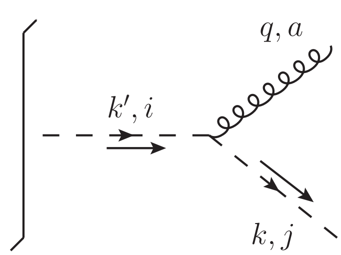
$$= (-ig\mu^\varepsilon)\,(k'+k)^\mu\, T^a_{ij}\,\varepsilon_\mu(q) \tag{71}$$

where a, i, j are color indices. Taking a Hermitian/complex conjugate gives

$$\mathcal{M}^*_{sL} = (ig\mu^\varepsilon)\,(k'+k)^\mu\, T^{a*}_{ij}\,\varepsilon^*_\mu(q) = (ig\mu^\varepsilon)\,(k'+k)^\mu\, T^a_{ji}\,\varepsilon^*_\mu(q) \tag{72}$$

to be understood as the inverse process $\widetilde{q}_j(k') \to \widetilde{q}_i(k) + g_a(q)$ with i taken to be $-i$, i.e.,

$\mathcal{M}^*_{sL} =$

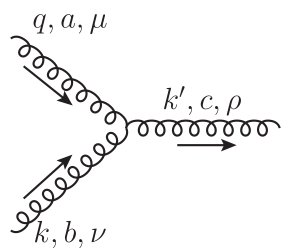


$$\equiv \mathcal{M}_{sR} \tag{73}$$

[9] By "initial" and "final" states, we are refering to the simple process like in Eq. (65) where only three particles are involved. The exact meaning is of (bc) ordering is the order we write down the Feynman amplitude.

This still follows the rule stated above. However, for tri-gluon vertex, this is more complex. The process process $g_a(q) + g_b(k) \to g_c(k')$ has the amplitude

$\mathcal{M}_{gL} =$



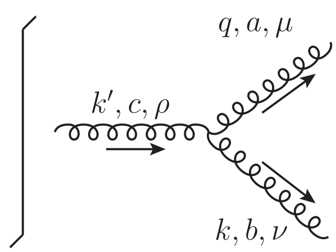
$$= -g f^{abc} \left[(q-k)^\rho g^{\mu\nu} + (k+k')^\mu g^{\nu\rho} + (-k'-q)^\nu g^{\rho\mu}\right] \quad (74)$$

where we have omitted the polarization vectors and the μ^ε factor. It has the structure that can be schemetically written as

$$f^{abc}\left[(a-b)^c + (b-c)^a + (c-a)^b\right] \quad (75)$$

Both f^{abc} and the momentum tensor are cyclic and antisymmtric with any two elements. So it does not matter whether we write clockwise or counter-clockwise, or which leg we start with. But it does matter when we reverse all the momenta, which gives a minus sign. On the one hand, the complex conjugate of $\mathcal{M}_{gL}$ is itself. On the other hand, when we take its inverse process and reverse all the momenta, as is conventional for a cut diagram, we result in

$\mathcal{M}_{gR} = \mathcal{M}_{gL}^* =$



$$\begin{aligned} &= -g f^{abc} \left[(q-k)^\rho g^{\mu\nu} + (k+k')^\mu g^{\nu\rho} + (-k'-q)^\nu g^{\rho\mu}\right] \\ &= g f^{abc} \left[(-q+k)^\rho g^{\mu\nu} + (-k-k')^\mu g^{\nu\rho} + (k'+q)^\nu g^{\rho\mu}\right] \end{aligned} \quad (76)$$

where we have written the amplitude to the right of the cut line in the usual Feynman rules, which causes the *extra minus sign.*

The complication of trigluon coupling is because the three gluons are identical so that there is no way to tell which one is the "gauge boson" and which ones are the "colored particles" engaging in the strong interactions in the adjoint representation. In fact, the trigluon coupling is a cyclic version of the scalar QCD coupling. For example, in Eq. (74) we can first regard (q, a, μ) as the gauge boson and the other two as the colored particles in adjoint representations, which gives us the coupling

$$\mathcal{M}_{gL} \supset (-ig)(-if^{acb})(k'+k)^{\mu}(-g^{\rho\nu}) \tag{77}$$

which resembles the scalar QCD interaction with just an extra $(-g^{\rho\nu})$ factor indicating the vector property of the gluons. By cycling (abc), we can get the full expression in Eq. (74). At this stage, we can understand the difference made by f^{abc}. Although f^{abc} is real and does not change by complex conjugation, it is $(-if^{abc}) = (T^a_A)_{bc}$, the adjoint representation, that corresponds to the Hermitian generator T_{ij} in other representations. By the same rule when taking complex conjugate, we should also reverse i and j, and here we need to reverse b and c, i.e.,

$$-gf^{abc} = (-ig)(-if^{abc}) = (-ig)(T^a_A)_{bc} \xrightarrow[conjugate]{complex} (ig)(T^a_A)_{cb} = gf^{acb} \tag{78}$$

The order of (acb) corresponds more naturally to the reversed diagram. So the minus sign comes from the same origin as the previous quark and scalar case, by reversing the sign of i by $-i$ in the vertex. It is just that the i is hidden in writting f^{abc} explicitly. So we add a rule that when there is a f^{abc} factor in the coupling, we add a minus sign. This does not affect the four-gluon coupling since there we have an explicit i factor and two f^{abc}, so the net effect is only replace i by $-i$.

The above discussion is sufficient for the cut rules on the right of the cut line. But for PDF, we need to prove another thing. Because the cut diagram for PDF is not a genuine amplitude, but has explicit spinor indices for both sides of the cut line, i.e., $M_L = \langle X|\psi_\alpha(0)|P\rangle$ has a spinor index α, and $M_R = \langle P|\bar{\psi}_\beta(0)|X\rangle$ has a spinor index β. So the spinor structure of M_L can be written as

$$M_L \sim c\,\gamma^{\mu_1}\cdots\gamma^{\mu_n}u \tag{79}$$

Its Hermitian conjugate is

$$M_L^\dagger = \langle P|\psi_\alpha^\dagger(0)|X\rangle \sim c^*\,u^\dagger\gamma^{\mu_n\dagger}\cdots\gamma^{\mu_1\dagger} \tag{80}$$

The M_R is actually defined using $\bar{\psi}$, which adds an additional γ^0 to $M_L^\dagger$, giving

$$M_R = M_L^\dagger\gamma^0 = \langle P|\bar{\psi}_\alpha(0)|X\rangle \sim c^*\,u^\dagger\gamma^{\mu_n\dagger}\cdots\gamma^{\mu_1\dagger}\gamma^0 = c^*\,\bar{u}\gamma^{\mu_n}\cdots\gamma^{\mu_1} \tag{81}$$

giving exactly the structure we need for the reversed process of the left side of the cut line.

This also leads to a useful property of cut diagram. When we flip the left side M_L on the cut line to the right, we will flip all the momentum flow, and the right part M_R obtained in this way is

$$M_R = M_L^\dagger \gamma^0 \tag{82}$$

If we flip M_R to the left, we get

$$M_L = \gamma^0 M_R^\dagger \tag{83}$$

so if we reverse the whole diagram, we obtain

$$M_R M_L \xrightarrow{flip} M_L^\dagger \gamma^0 \gamma^0 M_R^\dagger = (M_R M_L)^\dagger \tag{84}$$

that is, reversing a cut diagram gives its Hermitian conjugate!

b. Rules for the cut lines. The integration of k in Eq. (53) makes it look like a loop momentum. This is not exactly true, because even in a diagram without any loop we still need to integrate over k. When there are loops, k will entangle with the loop momenta and make it complicated. It is worth making this clear.

According to our definition of the cut line in Eq. (26), physical states X flow across the cut with physical on-shell momenta, where "physical" means with positive + component. When X contains $N \geq 1$ particles, with momenta $q_1, \cdots, q_N$, the sum over X contains a phase space integration

$$\int \prod_{i=1}^{N} \frac{d^{d-1}\boldsymbol{q_i}}{(2\pi)^{d-1} 2E_{q_i}} = \int \prod_{i=1}^{N} \frac{d^d q_i}{(2\pi)^d} (2\pi)\, \delta(q_i^2 - m_i^2) \theta(q_i^+) \tag{85}$$

where we have taken q_i to go from the left of the cut to the right. Together with the k integration and the momentum conservation δ-function, we have

$$\int \frac{d^d k}{(2\pi)^d} \delta(k^+ - xP^+) \int \prod_{i=1}^{N} \frac{d^d q_i}{(2\pi)^d} (2\pi)\, \delta(q_i^2 - m_i^2) \theta(q_i^+) (2\pi)^d \delta^d(P - \sum_{i=1}^{N} q_i - k) \tag{86}$$

Integrating out k using the δ-function results in

$$\int \prod_{i=1}^{N} \frac{d^d q_i}{(2\pi)^d} (2\pi)\, \delta(q_i^2 - m_i^2) \theta(q_i^+) \delta(P^+ - \sum_{i=1}^{N} q_i^+ - xP^+) \tag{87}$$

In this way, we treat each q_i as a loop momentum, but for the q_i line across the cut we associate with a factor $(2\pi)\, \delta(q_i^2 - m_i^2)\theta(q_i^+)$. And overall there is a factor $\delta(P^+ - \sum_{i=1}^{N} q_i^+ - xP^+)$ constraining all the loop momentum q_i.

The sum over X states also contains a sum over the spins for particles with spin higher than 0. A fermion with q_i on the left of the cut corresponds to a final-state particle line with $\bar{u}(q_i)$. It has a corresponding $u(q_i)$ on the right of the cut. Then the spin sum gives

$$\sum_s u_s(q_i)\bar{u}_s(q_i) = (\not{q}_i + m_i) \tag{88}$$

If the fermion line flows from the right of the right to the left (but the momentum q_i still flows to the right), we have instead,

$$\sum_s v_s(q_i)\bar{v}_s(q_i) = (\not{q}_i - m_i) = -(-\not{q}_i + m_i) \tag{89}$$

where $-q_i$ is the momentum flows along the fermion line arrow. So if we want to assign factor $\not{q} + m$ to a cut fermion line with momentum q flows along the fermion line, we need to add an extra -1 when the fermion line goes from the right of the cut to the left. This means that if a fermion loop is cut, we would have an overall -1 for the loop.

If a gluon line with momentum q is cut, we would have

$$\sum_{\lambda=1,2} \epsilon_\mu^\lambda(q)\epsilon_\nu^{\lambda *}(q) \to -g_{\mu\nu} \tag{90}$$

where the sum is over physical polarizations and the replacement of the polarization vector sum by $-g_{\mu\nu}$ uses Ward identity. Ward identity still holds in the presence of the Wilson line.

When X contains $N = 0$ particle, i.e., $|X\rangle = |0\rangle$ is the vacuum state, then the diagram contains only LO or virtual corrections. In this case, the k integration together with the momentum conservation δ-function becomes

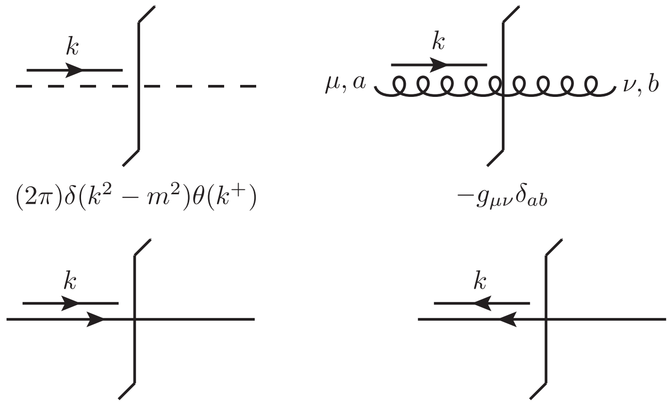
$$\int \frac{d^dk}{(2\pi)^d}\delta(k^+ - xP^+)(2\pi)^d\delta^d(P - k) = \delta(P^+ - xP^+) \tag{91}$$

Including the rule that the diagram on the right of the cut line is Hermitian conjugate of the one on the left, we have the following Feynman rules for cut diagrams

1. Usual Feynman rules apply to the left and right of the cut lines, but with all $i \to -i$ on the right parts. When the vertex involves a factor of f^{abc}, we add a -1 factor.

2. Each line across the cut line gives a momentum integration $\int d^dq/(2\pi)^d$.

3. There is an overall δ function $\delta(P^+ - \sum_{i=1}^N q_i^+ - xP^+)$. When no particle goes across the cut line ($N = 0$), it becomes $\delta(P^+ - xP^+)$.



$(2\pi)\delta(k^2 - m^2)\theta(k^+)$ $-g_{\mu\nu}\delta_{ab}$

$(2\pi)\delta(k^2 - m^2)\theta(k^+)\,(\gamma \cdot k + m)$ $-(2\pi)\delta(k^2 - m^2)\theta(-k^+)\,(\gamma \cdot k + m)$

FIG. 6: Feynman rules for cut propagaotrs. Dashed line is for scalar particle. a, b are color indices of gluons. Note that the momentum of fermion line flows along the fermion arrow, and there is an extra -1 factor for the fermion line crossing the cut line to the left.

4. For a cut propagator with momentum k (for fermions, k flows in the fermion line direction), replace the usual propagator by

$$\frac{i}{k^2 - m^2 + i\varepsilon} \to (2\pi)\delta(k^2 - m^2)\theta(k^+). \tag{92}$$

For a cut fermion propagator, add a factor $\not{k} + m$. For a gluon cut propagator, add a factor $-g_{\mu\nu}\delta_{ab}$, where a, b are the color indices.

5. When there is a fermion line going from the right of the cut to the left, add a factor -1.

The cut propagators are summarized in Fig. 6. Note that the second and third rules have already included the integral of k and the $\delta(k^+ - xP^+)$ factor in Eq. (53), so the parton vertices defined there should not include them.

5. More on cut diagrams.

We have been stressing that the diagram to the right of the cut line is the Hermitian conjugate of the left part. There may be puzzles, because the amplitude M_R on the right can be also expressed as usual matrix element, like in Eq. (24), and thus it seems to be an amplitude from $|X\rangle$ to $|P\rangle$, with the usual Feynman rules. How should we understand the rules summarized previously using that M_R is the complex ocnjugate of M_L?

This puzzle can be resolved if we are careful about the states. An amplitude is the matrix elements of an time-ordered operator between an "out" asymptotic state and an "in" asymptotic state. The DIS amplitude should be written as

$$M^\mu = \langle X; \text{out}|J^\mu(0)|P; \text{in}\rangle \tag{93}$$

where only write the part attached to the virtual photon. The DIS structure function is then

$$\begin{aligned}
W^{\mu\nu} &= \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(q+P-P_X)\, M^{\mu\dagger}(0)M^\nu(0) \\
&= \frac{1}{4\pi}\sum_X \int d^4x e^{i(q+P-P_X)\cdot x} \langle P; \text{in}|J^{\mu\dagger}(0)|X; \text{out}\rangle \langle X; \text{out}|J^\nu(0)|P; \text{in}\rangle \\
&= \frac{1}{4\pi}\sum_X \int d^4x e^{iq\cdot x}\langle P; \text{in}|J^{\mu\dagger}(x)|X; \text{out}\rangle \langle X; \text{out}|J^\nu(0)|P; \text{in}\rangle \\
&= \frac{1}{4\pi}\int d^4x e^{iq\cdot x}\langle P; \text{in}|J^{\mu\dagger}(x)\left\{\sum_X |X; \text{out}\rangle\langle X; \text{out}|\right\} J^\nu(0)|P; \text{in}\rangle \\
&= \frac{1}{4\pi}\int d^4x e^{iq\cdot x}\langle P; \text{in}|\left\{J^{\mu\dagger}(x)J^\nu(0)\right\}|P; \text{in}\rangle
\end{aligned} \tag{94}$$

where we used

$$\hat{P}^\mu|p; \text{in/out}\rangle = p^\mu|p; \text{in/out}\rangle \tag{95}$$

in going from the second line to third line, and

$$\sum_X |X; \text{out}\rangle\langle X; \text{out}| = \sum_X |X; \text{in}\rangle\langle X; \text{in}| = 1. \tag{96}$$

It is necessary that we are only using "X_{out}" states because only then do we have the completeness relation. Therefore the DIS structure function is the matrix element of normal ordered current-current operator between two "in" hadron states.

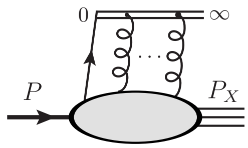
This also indicate that when factorizing $W^{\mu\nu}$ into PDF, the latter is also a matrix element of PDF operator between two "in" states

$$f_{i/P}(x) = \int_{-\infty}^{\infty}\frac{dw^-}{2\pi}e^{-ixP^+w^-}\left\langle P; \text{in}\left|\overline{\psi}_i(0^+, w^-, \mathbf{0}_T)\frac{\gamma^+}{2}\, W_F[w^-,\, 0]\,\psi_i(0)\right| P; \text{in}\right\rangle \tag{97a}$$

$$f_{g/P}(x) = \int_{-\infty}^{\infty}\frac{dw^-}{2\pi x P^+}e^{-ixP^+w^-}\left\langle P; \text{in}\left|G^{+j}(0^+, w^-, \mathbf{0}_T)\, W_A[w^-, 0]\, G^{+j}(0)\right| P; \text{in}\right\rangle \tag{97b}$$

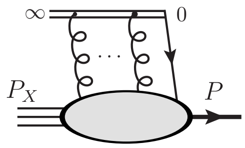
When we translate this into cut diagrams, we insert a complete set of states $\sum_X |X;\text{out}\rangle\langle X;\text{out}|$, and Eq. (23) and Eq. (24) become

$$M_L =$$



$$= \langle X;\text{out}\,| W_F^n[\infty,\,0]\,\psi_i(0)|\,P;\text{in}\rangle \tag{98}$$

$$M_R =$$



$$= \big\langle P;\text{in}\,\big|\overline{\psi}_i(0)W_F^n[0,\infty]\big|\,X;\text{out}\big\rangle = M_L^\dagger\gamma^0 \tag{99}$$

Therefore, only M_L can be interpreted as a Feynman amplitude, and M_R should be interpreted as its Hermitian conjugate, using the Feynman rules discussed in the previous subsection.

One may start worrying about the use of time-ordered operators as definitions of PDF in Eq. (39). This is not a problem because for a stable single particle, $|P;\text{in}\rangle = |P;\text{out}\rangle = |P\rangle$. So we are free to change the proton states such that

$$f_{i/P}(x) = \int_{-\infty}^{\infty}\frac{dw^-}{2\pi}e^{-ixP^+w^-}\left\langle P;\text{out}\left|T\left\{\overline{\psi}_i(0^+,w^-,\mathbf{0}_T)\frac{\gamma^+}{2}\,W_F[w^-,\,0]\,\psi_i(0)\right\}\right|P;\text{in}\right\rangle \tag{100a}$$

$$f_{g/P}(x) = \int_{-\infty}^{\infty}\frac{dw^-}{2\pi xP^+}e^{-ixP^+w^-}\left\langle P;\text{out}\left|T\left\{G^{+j}(0^+,w^-,\mathbf{0}_T)\,W_A[w^-,0]\,G^{+j}(0)\right\}\right|P;\text{in}\right\rangle \tag{100b}$$

Then they correspond exactly to time-ordered Feynman amplitudes. In contrast, in the cut-amplitude definition, it is better to use $\langle P;\text{in}|\cdots|P;\text{in}\rangle$ because the inserted states of X are both out states.

With this understanding, we can understand better the optical theorem. S matrix elements are defined as

$$S_{fi} = \langle f;\text{out}|i;\text{in}\rangle \tag{101}$$

where both "in" states and "out" states are orthoghonal, normalized, and complete: [10]

$$\langle f;\text{out}|f';\text{out}\rangle = \delta_{ff'},\quad \langle i;\text{in}|i';\text{in}\rangle = \delta_{ii'},\quad \sum_f |f;\text{out}\rangle\langle f;\text{out}| = \sum_i |i;\text{in}\rangle\langle i;\text{in}| = 1 \tag{102}$$

[10] Here $\delta_{ff'}$ is an abbreviate of the standard normalized δ function between states. For example, for single particle states, $\langle p;\text{out}|p';\text{out}\rangle = (2\pi)^3 2E_p\delta^3(\boldsymbol{p}-\boldsymbol{p}')$.

This makes the matrix S_{fi} a unitary matrix

$$\sum_i S_{fi}S^*_{f'i} = \delta_{ff'}, \quad \sum_f S_{fi}S^*_{fi'} = \delta_{ii'} \tag{103}$$

From this we can define a unitary operator $\hat{S}$ such that

$$S_{fi} = \langle f;\text{out}|i;\text{in}\rangle = \langle f;\text{out}|\hat{S}|i;\text{out}\rangle = \langle f;\text{in}|\hat{S}|i;\text{in}\rangle \tag{104}$$

or,

$$\hat{S}|i;\text{out}\rangle = |i;\text{in}\rangle, \quad \langle f;\text{in}|\hat{S} = \langle f;\text{out}| \tag{105}$$

The Feynman amplitude $i\mathcal{M}_{fi}$ is defined through

$$\begin{aligned} &\hat{S} = 1 + i\hat{\mathcal{T}}, \\ &S_{fi} = \langle f;\text{out}|\hat{S}|i;\text{out}\rangle = \langle f;\text{out}|1 + i\hat{\mathcal{T}}|i;\text{out}\rangle = \delta_{fi} + (2\pi)^4\delta^4(p_f - p_i)\cdot i\mathcal{M}_{fi} \end{aligned} \tag{106}$$

where the second line can be equally expressed in terms of "in" states. So we have

$$\langle f;\text{out}|1 + i\hat{\mathcal{T}}|i;\text{out}\rangle = \langle f;\text{in}|1 + i\hat{\mathcal{T}}|i;\text{in}\rangle = (2\pi)^4\delta^4(p_f - p_i)\cdot i\mathcal{M}_{fi} \tag{107}$$

The optical theorem follows from unitarity of $\hat{S}$ and a completeness relation (which is also an aspect of unitarity)

$$1 = \hat{S}^\dagger\hat{S} = (1 - i\mathcal{T}^\dagger)(1 + i\mathcal{T}) = 1 + i(\mathcal{T} - \mathcal{T}^\dagger) + \mathcal{T}^\dagger\mathcal{T} \tag{108}$$

so

$$i(\mathcal{T} - \mathcal{T}^\dagger) = -\mathcal{T}^\dagger\mathcal{T}. \tag{109}$$

Sandwiching between $\langle f;\text{in}|$ and $|i;\text{in}\rangle$ gives

$$\langle f;\text{in}|\, i(\mathcal{T} - \mathcal{T}^\dagger)\, |i;\text{in}\rangle = -\langle f;\text{in}|\, \mathcal{T}^\dagger\mathcal{T}\, |i;\text{in}\rangle \tag{110}$$

The left-hand side is

$$(2\pi)^4\delta^4(p_f - p_i)\left(i\mathcal{M}_{fi} - i\mathcal{M}^*_{if}\right) \tag{111}$$

and the right-hand side is, by inserting a complete set,

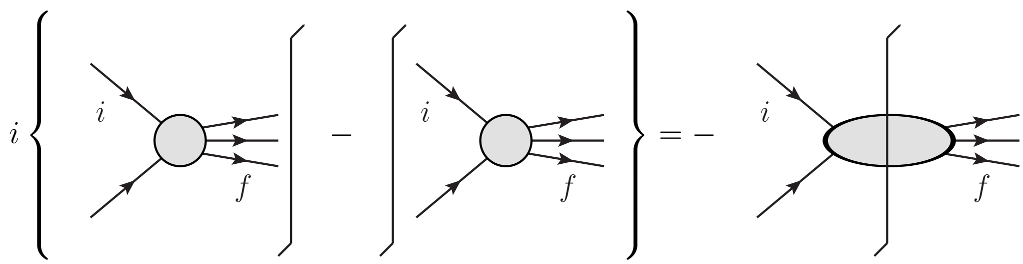
$$\begin{aligned} -\sum_X \langle f;\text{in}|\,\mathcal{T}^\dagger|X;\text{in}\rangle\langle X;\text{in}|\mathcal{T}\,|i;\text{in}\rangle &= -\sum_X \left[(2\pi)^4\delta^4(p_f - p_X)\mathcal{M}^*_{Xf}\right]\left[(2\pi)^4\delta^4(p_i - p_X)\mathcal{M}_{Xi}\right] \\ &= -(2\pi)^4\delta^4(p_f - p_i)\sum_X (2\pi)^4\delta^4(p_i - p_X)\left[\mathcal{M}^*_{Xf}\mathcal{M}_{Xi}\right] \end{aligned} \tag{112}$$

Thus, unitarity implies

$$i\mathcal{M}_{fi} - i\mathcal{M}^*_{if} = -\sum_X (2\pi)^4 \delta^4(p_i - p_X)\mathcal{M}^*_{Xf}\mathcal{M}_{Xi} \tag{113}$$

Although it is $i\mathcal{M}_{fi}$ that corresponds directly to a diagram, usually we we would factor the i out and just talk about $\mathcal{M}_{fi}$, because the LO diagram is usually pure imaginary (e.g., $\lambda\phi^4$ theory). Thus we will talk about the imaginary part of $\mathcal{M}_{fi}$, instead of the real part of $i\mathcal{M}_{fi}$, which is more of a convention issue.

$\mathcal{M}_{fi}$ is the amplitude going from initial state i to final state f. $\mathcal{M}^*_{if}$ is the complex conjugate of the amplitude $\mathcal{M}_{if} = \mathcal{M}_{i\leftarrow f}$. Using the cut-diagram language, taking a conjugate renders it an amplitude of $(i \to f)$, but to the right of the cut line. Similarly, $\mathcal{M}_{Xi}$ is the amplitude of $(i \to X)$ to the left the cut but $\mathcal{M}^*_{Xf}$ is the amplitude of $(X \to f)$ to the right the cut. In terms of diagrams, we have [11]

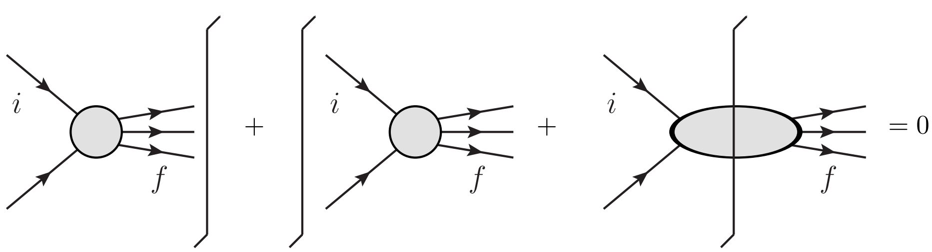


(115)

When $i = f$, Eq. (113) becomes

$$2\mathrm{Im}\mathcal{M}_{AA} = \sum_X (2\pi)^4 \delta^4(p_A - p_X)\mathcal{M}^*_{XA}\mathcal{M}_{XA} = \sum_X (2\pi)^4 \delta^4(p_A - p_X)|\mathcal{M}_{XA}|^2 \tag{116}$$

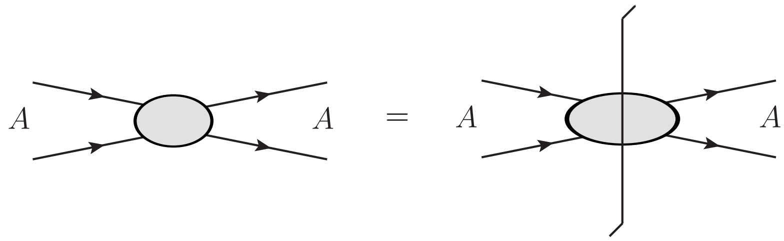
[11] If we do not follow the convention of factoring out the i factor, this can be expressed in a simpler form



(114)

which can be interpreted as that all possible cuts of a diagram (including the two cuts of putting the whole diagram to the left or right of the cut line) sum up to 0.

which can be expressed as

$$2\,\mathrm{Im}$$  (117)

where the cut line has the same meaning as (26), and the cut rules are the same as discussed in the previous subsection, except that there is no complication caused by the integration of k and the $\delta(k^+ - xP^+)$ function.

IV. CALCULATION OF PERTURBATIVE PDFS IN FEYNMAN GAUGE

In the preceding discussion we have been stressing the gauge invariance of the PDF definitions, which leads to the discussion of Wilson lines. Nevertheless, when going down to calculation, we need to specify a gauge to be able to write the gluon propagator. For this, we choose the covariant Feynman gauge. We can also choose light-cone gauge, where there is no gauge link at all. We will do that later.

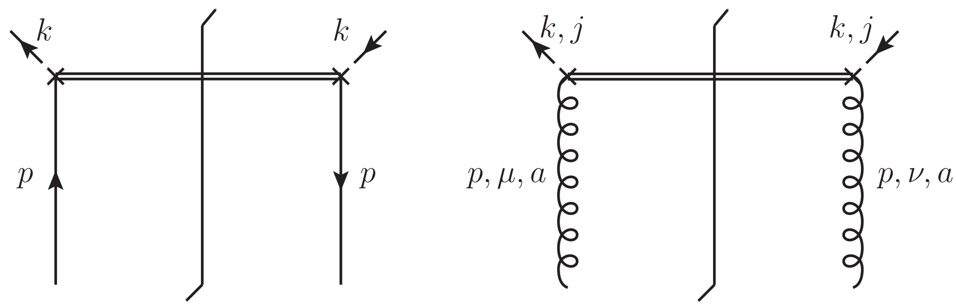
In the perturbative calculation, we take an on-shell parton as the target. Similar to the case in Eq. (1), we need to calculate the diagonal matrix element, i.e., the initial and final partons should have the same momentum, spin/polarization, and color. We can either specify a spin or color state, or average over them. Note that when averaging over on-shell gluon polarization states, we need to include only physical polarizations, which have $d - 2 = 2 - 2\varepsilon$ in d-dimensions. Throughout the calculations, we take partons as being massless.

A. LO PDFs

The LO diagrams for quark and gluon PDFs are shown in Fig. 7. Only these two diagrams can be drawn, so there are only $f^{[0]}_{q/q}(x)$ and $f^{[0]}_{g/g}(x)$.

$$f^{[0]}_{q/g}(x) = f^{[0]}_{g/q}(x) = 0 \tag{118}$$

Since no momentum flows along the Wilson line, we simply have $k = p$.



(a) (b)

FIG. 7: LO cut diagrams for quark and gluon PDFs.

For quark PDF, we have

$$f_{q/q}^{[0]}(x) = \delta(p^+ - xp^+)\mathrm{Tr}\left[\frac{\gamma^+}{2}\frac{\not{p}}{2}\right] = \delta(1-x) \tag{119}$$

where $\delta(p^+ - xp^+)$ is from the third rule listed on p. 24, $\gamma^+/2$ is the parton vertex in Eq. (53), and $\not{p}/2$ is the spin average of the spinors. The initial and final external quarks have the same color which has been averaged. Note that we can also choose not to average over the spins, but specify a certain spin s, which gives

$$f_{q/q}^{[0]}(x) = \delta(p^+ - xp^+)\left[\bar{u}_s(p)\frac{\gamma^+}{2}u_s(p)\right] = \delta(p^+ - xp^+)p^+ = \delta(1-x) \tag{120}$$

where we used $\bar{u}_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu \delta_{ss'}$.

For gluon PDF, we have

$$f_{g/g}^{[0]}(x) = \frac{1}{2-2\varepsilon}\sum_\lambda \delta(p^+ - xp^+)\frac{1}{xp^+}\cdot(-i)\left(p^+ g_\mu^j - p^j n_\mu\right)\varepsilon_\lambda^\mu(p)\cdot i\left(p^+ g_\nu^j - p^j n_\nu\right)\varepsilon_\lambda^{\nu *}(p) \tag{121}$$

where $\delta(p^+ - xp^+)$ is from the third rule listed on p. 24, $1/xp^+$ is the parton vertex in Eq. (53) and $\delta_{jj'}$ has been implicitly included as a sum over j (transverse index), $(-i)\left(p^+ g_\mu^j - p^j n_\mu\right)$ and $i\left(p^+ g_\nu^j - p^j n_\nu\right)$ are the gluon-gauge-link vertices in Fig. 2 where we have used $k = p$ to substitute p for k. λ denotes the gluon polarization and has been averaged. We choose $p = (p^+, 0^-, \mathbf{0}_T)$, and physical polarizations to satisfy

$$\varepsilon_\lambda^\pm = 0, \quad \varepsilon_\lambda \cdot \varepsilon_{\lambda'}^* = -\varepsilon_\lambda^j \cdot \varepsilon_{\lambda'}^{j*} = -\delta_{\lambda\lambda'}, \quad \sum_\lambda \varepsilon_\lambda^\mu(p)\varepsilon_\lambda^{\nu *}(p) = -g^{\mu\nu} + n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu \equiv -g_\perp^{\mu\nu} \tag{122}$$

Note that we have three ways to do it:

1. *Direct calculation for certain polarization.* Now we delete the $1/(2-2\varepsilon)\sum_\lambda$ and use a specified polarization state λ. Since $n\cdot\varepsilon=\varepsilon^+=0$, we have

$$f_{g/g}^{[0]}(x)=\delta(p^+-xp^+)\frac{1}{xp^+}\cdot p^+\varepsilon_\lambda^j(p)\cdot p^+\varepsilon_\lambda^{j*}(p)=\delta(1-x) \tag{123}$$

2. *Polarization average using Eq.* (122). Eq. (122) represents the exact results for physical gluon polarization vectors. Sum over λ gives

$$\begin{aligned} f_{g/g}^{[0]}(x)&=\frac{1}{2-2\varepsilon}\delta(p^+-xp^+)\frac{1}{xp^+}\cdot(-i)\left(p^+g_\mu^j-p^jn_\mu\right)\cdot i\left(p^+g_\nu^j-p^jn_\nu\right)\left(-g_\perp^{\mu\nu}\right)\\ &=\frac{1}{2-2\varepsilon}\delta(p^+-xp^+)\frac{1}{xp^+}\cdot\left(p^+\right)^2\left(-g^{jj}\right)=\delta(1-x) \end{aligned} \tag{124}$$

where in the second line we used $g_\perp^{\mu\nu}n_\mu=g_\perp^{\mu\nu}n_\nu=0$ and $g^{jj}=-(2-2\varepsilon)$.

3. *Use of Ward identity.* It is easy to verify Ward identity by replacing $\varepsilon_\lambda^\mu(p)$ by p^μ, which gives

$$\left(p^+g_\mu^j-p^jn_\mu\right)p^\mu=p^+p^j-p^jn\cdot p=0 \tag{125}$$

This allows us to replace the polarization sum by

$$\sum_\lambda\varepsilon_\lambda^\mu(p)\varepsilon_\lambda^{\nu*}(p)\to-g^{\mu\nu} \tag{126}$$

And then

$$f_{g/g}^{[0]}(x)=\frac{1}{2-2\varepsilon}\delta(p^+-xp^+)\frac{1}{xp^+}\cdot\left(p^+g_\mu^j-p^jn_\mu\right)\cdot\left(p^+g_\nu^j-p^jn_\nu\right)\left(-g^{\mu\nu}\right)=\delta(1-x) \tag{127}$$

Note that although using $-g^{\mu\nu}$ makes it seem that we are using d polarization, we should still divide by $2-2\varepsilon$, the number of physical polarizations.

The above results can be summarized as

$$f_{j/k}^{[0]}(x)=\delta_{jk}\delta(1-x) \tag{128}$$

The LO calculations are simple, and we have illustrated different treatments of target spin states and verified that they are equivalent. For the NLO calculations, we will exclusively average over the spins (and colors) of the target.

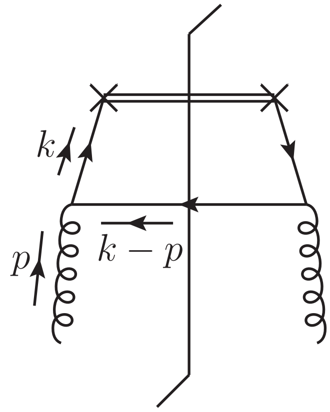


FIG. 8: NLO cut diagrams for quark PDF in a gluon target.

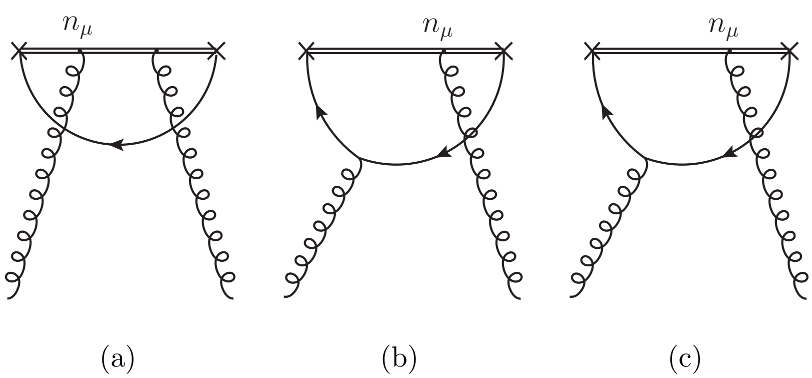


FIG. 9: The other three NLO cut diagrams for quark PDF in a gluon target where g legs attach to the Wilson line. These diagrams vanish because $n \cdot \varepsilon = 0$.

B. NLO: Quark in gluon

There is only one diagram shown in Fig. 8 and 9. The diagrams in Fig. 9 all contain gluon legs attached to the Wilson line, which gives $n \cdot \varepsilon = \varepsilon^+ = 0$ due to our choice in Eq. (122). Thus the only contribution comes from Fig. 8, where we have used the fact that no momentum flows along the Wilson line to write the momentum of the quark propagator as k.

Using the Feynman rules summarized before, we have

$$
\begin{aligned}
f^{[1]}_{(0)q/g}(x) &= -\int \frac{d^dk}{(2\pi)^d}\delta(k^+ - xp^+)\mathrm{Tr}\left[\frac{\gamma^+}{2}\frac{i\not{k}}{k^2+i\varepsilon}\left(-ig\mu^\varepsilon \not{\epsilon}(p)T^a_{ij}\right)\times\right. \\
&\qquad \left.\times(\not{k}-\not{p})\,(2\pi)\delta\left((k-p)^2\right)\theta(p^+-k^+)\left(ig\mu^\varepsilon\not{\epsilon}^*(p)T^a_{ji}\right)\frac{-i\not{k}}{k^2-i\varepsilon}\right] \\
&= -g^2\mu^{2\varepsilon}T_F\theta(1-x)\int\frac{dk^-d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^d}(2\pi)\delta\left((k-p)^2\right)\frac{\mathrm{Tr}\left[\frac{\gamma^+}{2}\not{k}\,\not{\epsilon}(p)\,(\not{k}-\not{p})\,\not{\epsilon}^*(p)\,\not{k}\right]}{(k^2+i\varepsilon)(k^2-i\varepsilon)}
\end{aligned}
\tag{129}
$$

where i, j are the colors of the fermions and are summed over, but the gluon color a is not, which gives T_F. The subscript (0) means that this is bare PDF, which contains UV divergence and is subject to renormalization. In the first line, we equivalently write the integration of q in rule 2 on p. 24 as the integration over k, and the overall minus sign is because the cut fermion propagator flows to the left of the cut line. In the second line we have used the $\delta(k^+ - xp^+)$ function to integrate out k^+. $\delta\left((k-p)^2\right)$ is the on-shell condition for the antiquark (the cut propagator). We can use it to integrate out k^-.

$$
(k-p)^2 = 2(k-p)^+(k-p)^- - k_T^2 = 2(x-1)p^+k^- - k_T^2 = -2(1-x)p^+\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right)
$$

$$
\delta\left((k-p)^2\right) = \frac{1}{2(1-x)p^+}\delta\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right) \tag{130}
$$

This fixes $k^- = -k_T^2/2(1-x)p^+ < 0$, and thus we have

$$
k = \left(xp^+,\ -\frac{k_T^2}{2(1-x)p^+},\ \boldsymbol{k}_T\right) \tag{131}
$$

$$
(p-k) = \left((1-x)p^+,\ \frac{k_T^2}{2(1-x)p^+},\ -\boldsymbol{k}_T\right) \tag{132}
$$

as the momentum for the quark and antiquark split by the gluon with on-shell momentum $p = (p^+, 0^-, \mathbf{0}_T)$. This is an important feature of parton density — the parton we are seeking is *off-shell*, and the target/mother parton and the other daughter parton are on shell, and the phase space of the other daughter parton is integrated over. The off-shellness of the parton is characterized by k_T:

$$
k^2 = 2xp^+\cdot\left(-\frac{k_T^2}{2(1-x)p^+}\right) - k_T^2 = -\frac{k_T^2}{1-x} \tag{133}
$$

which is space-like. The smaller k_T is, the closer the parton is to be on-shell and the more collinear the two partons are. For small k_T, we can connect it the angle θ at which the parton is split out

$$\theta \simeq \frac{k_T}{k^z} \simeq \frac{\sqrt{2}k_T}{xp^+} \tag{134}$$

This can be connected to the parton shower picture in the initial state radiation where a parton comes out of the proton in a collision and keeps radiating (away on-shell partons) and develops a high k_T (virtuality). This picture is in contrast to the jet picture, or the final state radiation, where an off-shell parton goes out from the hard collision and keeps radiating (away on-shell partons) until its k_T (virtuality) becomes very small and hadronizes. There the NLO parton picture would be an off-shell parton splitting into two on-shell partons.

Using the δ function to integrate out k^-, we now calculate the trace in Eq. (129). First we average the polarization of the gluon, which gives

$$\mathrm{Tr}\left[\frac{\gamma^+}{2}\, \not{k}\, \not{\epsilon}(p)\, (\not{k}-\not{p})\, \not{\epsilon}^*(p)\, \not{k}\right] \xrightarrow[\text{polari.}]{\text{average}} \frac{-1}{2-2\varepsilon}\mathrm{Tr}\left[\frac{\gamma^+}{2}\, \not{k}\, \gamma^\mu\, (\not{k}-\not{p})\, \gamma^\nu\, \not{k}\right](g_{\mu\nu} - n_\mu \bar{n}_\nu - n_\nu \bar{n}_\mu) \tag{135}$$

$g_{\mu\nu}$ term gives

$$\begin{aligned} &g_{\mu\nu}\mathrm{Tr}\left[\frac{\gamma^+}{2}\, \not{k}\, \gamma^\mu\, (\not{k}-\not{p})\, \gamma^\nu\, \not{k}\right] = \frac{2-d}{2}\mathrm{Tr}\left[\gamma^+\, \not{k}\, (\not{k}-\not{p})\, \not{k}\right] \\ =& -4(1-\varepsilon)\left[2k^+\, k\cdot(k-p) - (k-p)^+\, k^2\right] = \frac{4(1-\varepsilon)}{1-x}p^+\, k_T^2 \end{aligned} \tag{136}$$

$n_\mu \bar{n}_\nu$ term gives

$$\begin{aligned} &n_\mu \bar{n}_\nu \mathrm{Tr}\left[\frac{\gamma^+}{2}\, \not{k}\, \gamma^\mu\, (\not{k}-\not{p})\, \gamma^\nu\, \not{k}\right] \\ =&\frac{1}{2}\mathrm{Tr}\left[\gamma^+\, \not{k}\, \gamma^+\, (\not{k}-\not{p})\, \gamma^-\, \not{k}\right] = k^+\mathrm{Tr}\left[\gamma^+\, (\not{k}-\not{p})\, \gamma^-\, \not{k}\right] \\ =&4k^+\left[(k-p)^+ k^- - k\cdot(k-p) + k^+(k-p)^-\right] = 4xp^+k_T^2 \end{aligned} \tag{137}$$

and $n_\nu \bar{n}_\mu$ term gives

$$\begin{aligned} &n_\nu \bar{n}_\mu \mathrm{Tr}\left[\frac{\gamma^+}{2}\, \not{k}\, \gamma^\mu\, (\not{k}-\not{p})\, \gamma^\nu\, \not{k}\right] \\ =&\frac{1}{2}\mathrm{Tr}\left[\gamma^+\, \not{k}\, \gamma^-\, (\not{k}-\not{p})\, \gamma^+\, \not{k}\right] = k^+\mathrm{Tr}\left[\gamma^+\, \not{k}\, \gamma^-\, (\not{k}-\not{p})\right] \\ =&4k^+\left[k^+(k-p)^- - k\cdot(k-p) + (k-p)^+ k^-\right] = 4xp^+k_T^2 \end{aligned} \tag{138}$$

where for $n_\mu \bar{n}_\nu$ and $n_\nu \bar{n}_\mu$ terms we have used $\gamma^+\gamma^+ = \gamma^-\gamma^- = 0$ and $g^{+-} = g^{-+} = 1$. Then the trace in Eq. (135) becomes

$$\mathrm{Tr}\left[\frac{\gamma^+}{2}\,\not{k}\,\not{\varepsilon}(p)\,(\not{k}-\not{p})\,\not{\varepsilon}^*(p)\,\not{k}\right]\xrightarrow[\text{polari.}]{\text{average}}\frac{-1}{2-2\varepsilon}\left[\frac{4(1-\varepsilon)}{1-x}p^+\,k_T^2-8xp^+k_T^2\right]$$
$$=-\frac{2p^+k_T^2}{1-x}\left[1-\frac{2x(1-x)}{1-\varepsilon}\right] \tag{139}$$

Note that since $n_\mu\bar{n}_\nu$ and $n_\nu\bar{n}_\mu$ terms are non-zero, we cannot replace the polarization sum by $-g_{\mu\nu}$, as we usually do in calculating Feynman diagrams. This is because we already used the transverse polarization condition to render the diagrams in Fig. 9 to zero, and therefore we have to use exact transverse polarization sum in calculating the diagram in Fig. 8. In other words, the diagram in Fig. 8 alone does not satisfy Ward identity. One has to include all those four diagrams to guarantee Ward identity. Only in that case can we safely do $\sum_\lambda \varepsilon_\lambda^\mu(p)\varepsilon_\lambda^{\nu*}(p)\to -g^{\mu\nu}$.

Plugging Eq. (130), (133) and (139) into Eq. (129) gives

$$f^{[1]}_{(0)q/g}(x)=g^2\mu^{2\varepsilon}T_F\theta(1-x)\int\frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\frac{1}{k_T^2}\left[1-\frac{2x(1-x)}{1-\varepsilon}\right] \tag{140}$$

Since the integrand is spherically symmetric with $\boldsymbol{k}_T$, we can use [1]

$$\int d^d\boldsymbol{k}f(\boldsymbol{k}^2)=\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty dk^2\,(k^2)^{d/2-1}\,f(k^2) \tag{141}$$

to write

$$\int\frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-2}}=\frac{(4\pi)^\varepsilon}{(4\pi)\Gamma(1-\varepsilon)}\int_0^\infty dk_T^2(k_T^2)^{-\varepsilon} \tag{142}$$

And then

$$f^{[1]}_{(0)q/g}(x)=\frac{g^2}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}T_F\int_0^\infty\frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon}\left[1-\frac{2x(1-x)}{1-\varepsilon}\right] \tag{143}$$

(where we have omitted $\theta(1-x)$ factor, as will be explained later.) We see that as $\varepsilon\to 0$, we have both UV divergence and collinear divergences. The UV divergence comes because the integration domain of k_T extends to infinity. That does not reflect the true physical reality and is a consequence of factorization approximation, which disentangles k_T integration from the hard process. If we do not perform factorization, then the hadron part (that becomes PDF after factorization) forms a closed loop with the hard part, and there is no UV divergence from k integral. Therefore, the renormalization of PDF is separate from QCD renormalization and should be regarded as an artifact. After removing UV divergence, we are left with collinear divergence. Collinear divergence arises from the configuration where the quark is collinear to the gluon (and the antiquark) (see Eq. (134)[12]), and is a result of on-shell and massless parton approximation. In reality, we do not have on-shell partons and quarks are not massless. So collinear divergence does not really exist in nature, but only reflects its nature of being long-distance origin. When calculating hard scattering coefficients, we need to systematically subtract these long-distance (collinear divergent in massless calculation) parts.

Dimensional regularization (DR) regulates both UV (by $\varepsilon > 0$) and IR (by $\varepsilon < 0$) divergences. A scaleless integral, like the k_T integral in Eq. (143), in DR gives 0. This is one way of saying that IR divergence cancels UV divergence, but this is very misleading because UV divergence is always canceled by UV counterterms, and the IR divergence is left. The result is that the renormalized $f^{[1]}_{q/g}(x)$ is equal to *the negative of the UV pole*. To extract the UV pole, we cutoff the lower limit of the k_T integration

$$\int_{\Lambda^2}^{\infty} \frac{dk_T^2}{k_T^2} (k_T^2)^{-\varepsilon} = \frac{1}{\varepsilon} (\Lambda^2)^{-\varepsilon} \tag{144}$$

where we have to use $\varepsilon > 0$. The UV pole is then $1/\varepsilon$. In $\overline{\text{MS}}$ scheme, we would write as $1/\varepsilon - \gamma_E + \ln 4\pi$, or $S_\varepsilon/\varepsilon$ Eq. (41). So we have

$$\text{UV pole of } \int_0^{\infty} \frac{dk_T^2}{k_T^2} (k_T^2)^{-\varepsilon} = \frac{S_\varepsilon}{\varepsilon} \tag{145}$$

And therefore,

$$\text{UV pole of } f^{[1]}_{(0)q/g}(x) = \frac{g^2}{8\pi^2} T_F \left[x^2 + (1-x)^2\right] \frac{S_\varepsilon}{\varepsilon} \tag{146}$$

From Eq. (46), the (UV) renormalized quark PDF in gluon target is

$$f^{[1]}_{q/g}(x) = f^{[1]}_{(0)q/g}(x) + Z^{[1]}_{qg}(x). \tag{147}$$

In $\overline{\text{MS}}$ scheme, $Z_{qg}(x)$ only removes the UV pole in $f^{[1]}_{(0)q/g}(x)$, so

$$Z^{[1]}_{qg}(x) = -\frac{g^2}{8\pi^2} T_F \left[x^2 + (1-x)^2\right] \frac{S_\varepsilon}{\varepsilon} \tag{148}$$

and then

$$\boxed{f^{[1]}_{q/g}(x) = 0 + Z^{[1]}_{qg}(x) = -\frac{g^2}{8\pi^2} T_F \left[x^2 + (1-x)^2\right] \frac{S_\varepsilon}{\varepsilon}} \tag{149}$$

[12] In 4-dimensions, the k_T integral can be written in terms of θ as $\int dk_T^2/k_T^2 = 2\int d\theta/\theta$, from which we can clearly see the logarithmic collinear divergence.

which is purely the IR pole. As we have explained, being IR divergent is not a big problem since $f^{[1]}_{q/g}(x)$ is not a real object in nature. It is more of a theory method to subtract long-distance piece from the hard coefficients, and moreover it gives the evolution kernel for RGE of the renormalized PDF. From Eq. (52) we have

$$\boxed{P^{[1]}_{qg}(z) = \frac{g^2}{8\pi^2} T_F \left[z^2 + (1-z)^2\right]} \tag{150}$$

Now let's explain the $\theta(1-x)$ factor omitted in Eq. (143). In fact, our results in Eq. (149) and (150) are only true when $x > 0$ (which is enough for our purpose). The $x < 0$ part can be always obtained using Eq. (38) where P can be any target, and in particular, the on-shell gluon target here, so we have when $x < 0$,

$$f^{[1]}_{q/g}(x) = -f^{[1]}_{\bar{q}/g}(-x) = \frac{g^2}{8\pi^2} T_F \theta(1+x) \left[x^2 + (1+x)^2\right] \frac{S_\varepsilon}{\varepsilon}. \tag{151}$$

where we have used the charge symmetry to write down the antiquark PDF in a gluon when $x > 0$

$$f^{[1]}_{\bar{q}/g}(x) = f^{[1]}_{q/g}(x) = -\frac{g^2}{8\pi^2} T_F \theta(1-x) \left[x^2 + (1-x)^2\right] \frac{S_\varepsilon}{\varepsilon} \tag{152}$$

We see that when $x < 0$, the $\theta(1+x)$ factor sets a lower bound -1 to x. Note that it is because gluon is charge neutral that have $f^{[1]}_{\bar{q}/g}(x) = f^{[1]}_{q/g}(x)$, but $f^{[1]}_{q/g}(x) = -f^{[1]}_{\bar{q}/g}(-x)$ is true for any target, not necessarily neutral. Then we have the complete quark PDF in a gluon

$$f^{[1]}_{q/g}(x) = \begin{cases} -\dfrac{g^2}{8\pi^2} T_F \left[x^2 + (1-x)^2\right] \dfrac{S_\varepsilon}{\varepsilon} & \text{when} 0 < x < 1 \\ +\dfrac{g^2}{8\pi^2} T_F \left[x^2 + (1+x)^2\right] \dfrac{S_\varepsilon}{\varepsilon} & \text{when } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases} \tag{153}$$

which is an odd function of x. This can also be calculated using the same approach, but for $x < 0$ there are more cut diagrams and the calculation is more involved. Since it does not turn out to be very useful, I will not discuss it unless I have time to complete this part later. And the antiquark PDF can be also provided.

C. NLO: Quark in quark

The diagrams for quark PDF in a (on-shell massless) quark target are shown in Fig. 10-12, where (a)(b)(g) should be accompanied by their Hermitian conjugate, obtained by flipping left and right. (d) and (f) are each other's Hermitian conjugate.

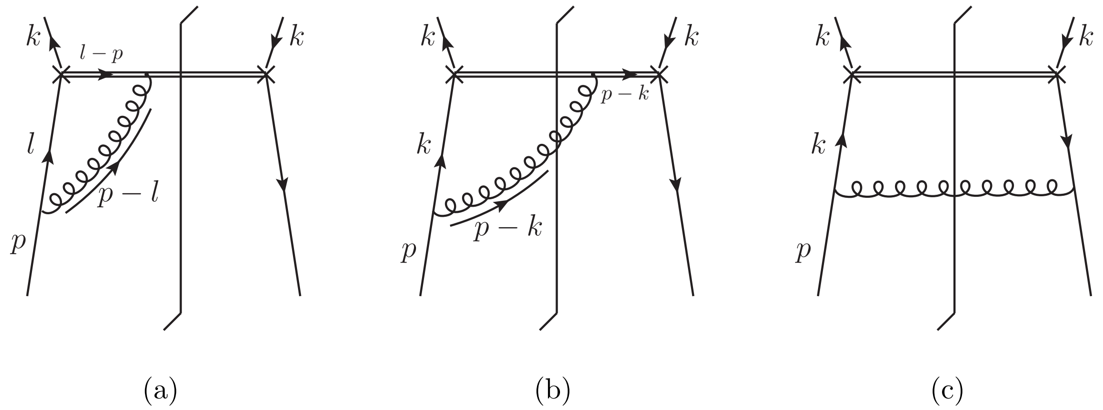


FIG. 10: Three of the NLO cut diagrams for quark PDF in a quark target. Note that no momentum flows through the Wilson line across the cut line. This allows to label the momentum on the Wilson line in (a) and (b). The Hermitian conjugate of diagrams (a) and (b) should also be included.

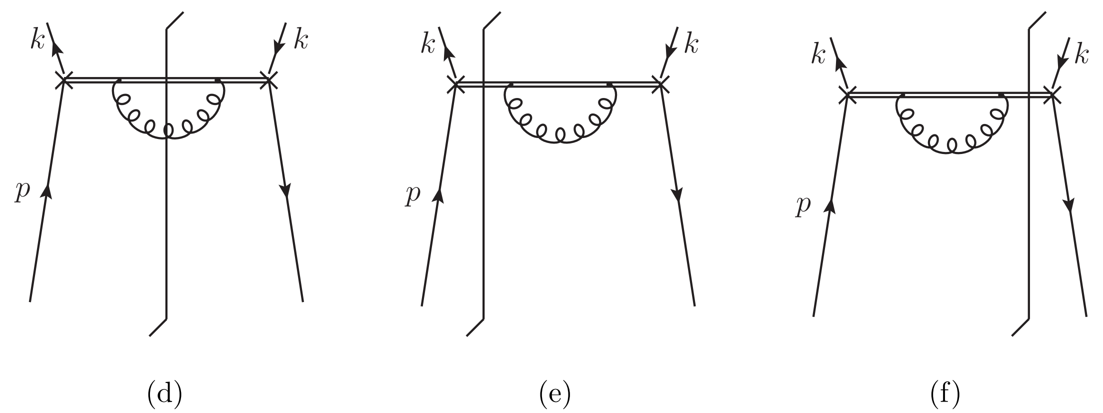
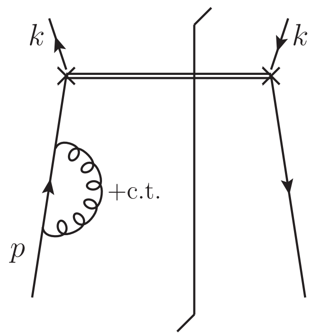


FIG. 11: Three of NLO cut diagrams for quark PDF in a quark target. These diagrams vanish because $n_\mu\, g^{\mu\nu} n_\nu = n^2 = 0$.

A gluon attached to the Wilson line has a vertex proportional to n_μ. A gluon propagator contains a $g^{\mu\nu}$ factor. If the both ends of a gluon propagator attach to the Wilson line, the diagram is proportional to $n_\mu\, g^{\mu\nu} n_\nu = n^2 = 0$. So diagrams (d)(e)(f) in Fig. 11 vanish.

1. *Virtual gluon radiation (a)*

Diagram (a) in Fig. 10 represents a virtual gluon radiation. In the DIS factorization context, it comes from the virtual collinear gluon radiation of the quark parton, and is factorized into the Wilson line. So this part is supposed to only capture the collinear configuration. But after factorized into the PDF, the loop momentum l extends to infinity,



(g)

FIG. 12: NLO external leg corrections for quark PDF in a quark target, accompanied by the QCD counterterm due to the field renormalization factor Z_2. The Hermitian conjugated diagram should also be included. Since their UV divergences have already been subtracted by Z_2, they do not contribute new UV divergences to the bare PDF.

and then the Wilson line propagator leads to a new divergence called *rapidity divergence*. This is not inherent in the DIS process, and we will see a cancelation of rapidity divergence between the virtual emission (a) and the real emission (b).

Following the Feynman rules outlined in Sec. III, diagram (a) together with its Hermitian conjugate gives

$$\begin{aligned} f_{(0)\,q/q}^{(a+a^\dagger)} &= \delta(p^+ - xp^+) \int \frac{d^d l}{(2\pi)^d} \times \\ &\quad \times \operatorname{Tr}\left[\frac{\gamma^+}{2}\left(-ig\mu^\varepsilon n_\mu T^a_{ij}\right) \frac{i}{n\cdot(l-p)+i\varepsilon}\frac{i\not{l}}{l^2+i\varepsilon}\left(-ig\mu^\varepsilon \gamma_\nu T^a_{ji}\right)\frac{-ig^{\mu\nu}}{(p-l)^2+i\varepsilon}\frac{\not{p}}{2}\right] + \text{h.c.} \\ &= -ig^2\mu^{2\varepsilon}C_F\delta(p^+ - xp^+)\int \frac{d^d l}{(2\pi)^d}\frac{\operatorname{Tr}\left[\frac{\gamma^+}{2}\not{l}\gamma^+\frac{\not{p}}{2}\right]}{(l^+ - p^+ + i\varepsilon)\,(l^2+i\varepsilon)\,((p-l)^2+i\varepsilon)} + \text{h.c.} \end{aligned} \tag{154}$$

where $\delta(p^+ - xp^+)$ is from rule 3 on p. 24 since no momentum flows to the final states, $\left(-ig\mu^\varepsilon n_\mu T^a_{ij}\right)$ is the Wilson line vertex for the attached gluon and T^a_{ij} is the fundamental representation generator with a being the gluon color index and i, j the quark color indices, and $i/\left(n\cdot(l-p)\right)$ is the gauge link propagator whose momentum is obtained from the momentum conservation of the link vertex. There is a sum over a and j, but not i, which gives the C_F factor.

l is a genuine loop momentum, each component extending to infinity. The gauge link propagator only contains l^+ component. This suggests to express the integration in light-

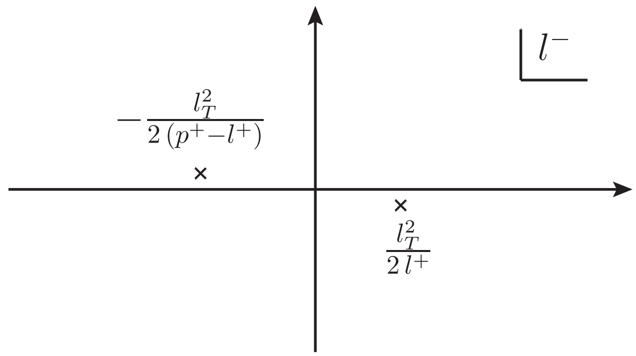


FIG. 13: The pole structure of l^- for the loop momentum l in the virtual correction of quark PDF in quark target (diagram (a) in Fig. 10), when $0 < l^+ < p^+$.

cone coordinate

$$\int d^d l = \int dl^+ dl^- d^{2-2\varepsilon} \boldsymbol{l}_T \tag{155}$$

$$l^2 + i\varepsilon = 2l^+ l^- - l_T^2 + i\varepsilon \tag{156}$$

$$(p-l)^2 + i\varepsilon = -2\,(p^+ - l^+)\,l^- - l_T^2 + i\varepsilon \tag{157}$$

For the integral of l^-, there are two poles, one from the quark propagator l^2, the other from the gluon propagator $(p-l)^2$. For the integral to be non-zero, it is necessary that these two poles lie on different sides of the real axis on the complex l^- plane. This constrains l^+ to be between 0 and p^+:

$$\text{Only when } 0 < l^+ < p^+ \text{ is the } l^- \text{ integral non-zero.} \tag{158}$$

Then l^2 gives a l^- pole at $\dfrac{l_T^2}{2l^+} - i\varepsilon$ and $(p-l)^2$ at $-\dfrac{l_T^2}{2(p^+ - l^+)} + i\varepsilon$, shown in Fig. 13. The distance between them is

$$\frac{l_T^2}{2}\left(\frac{1}{l^+} + \frac{1}{p^+ - l^+}\right) \tag{159}$$

When $l_T \to 0$, they become infinitely closer so that the l^- contour is pinched by these two poles, which means that there is no way to deform the contour to avoid them and the integration must encompass the region where the integrand is very large (infinite as $l_T \to 0$). A non-zero l_T regulates the pinch singularity. So the integral of l^- is a singular function of l_T when $l_T \to 0$. We perform the l^- integration by deform the contour to encircle the pole of $(p-l)^2$ on the upper half of the complex plane, which picks the residue of l^- at

$-l_T^2/2(p^+ - l^+) + i\varepsilon$. Equivalently this is done by replacing the propagator of the gluon by

$$\frac{i}{(p-l)^2 + i\varepsilon} \to 2\pi i \cdot \frac{i}{-2(p^+ - l^+)} \delta\left(l^- + \frac{l_T^2}{2(p^+ - l^+)}\right) = 2\pi\delta\left((p-l)^2\right) \tag{160}$$

just like cutting the gluon propagator. Picking this pole then gives

$$l^2 = 2l^+\left(-\frac{l_T^2}{2(p^+ - l^+)}\right) - l_T^2 = -\frac{p^+}{p^+ - l^+} l_T^2 \tag{161}$$

$$\mathrm{Tr}\left[\frac{\gamma^+}{2} \not{l} \gamma^+ \frac{\not{p}}{2}\right] = 2l^+ p^+ \tag{162}$$

and the integral in Eq. (154) becomes

$$\begin{aligned}
&\int \frac{dl^+ d^{2-2\varepsilon}\boldsymbol{l}_T}{(2\pi)^d} \frac{2l^+ p^+}{(l^+ - p^+)\left(-\frac{p^+}{p^+ - l^+} l_T^2\right)} \frac{2\pi i}{-2(p^+ - l^+)} \\
=& -i \int_0^{p^+} \frac{l^+ dl^+}{p^+ - l^+} \int \frac{d^{2-2\varepsilon}\boldsymbol{l}_T}{(2\pi)^{d-1}} \frac{1}{l_T^2} \\
=& -ip^+ \frac{(4\pi)^\varepsilon}{(8\pi^2)\Gamma(1-\varepsilon)} \int_0^1 \frac{\alpha\, d\alpha}{1-\alpha} \int \frac{dl_T^2}{l_T^2} \left(l_T^2\right)^{-2\varepsilon}
\end{aligned} \tag{163}$$

where on the third line we have defined $\alpha = l^+/p^+$. Notice that the integral of l^- gives a $1/l_T^2$ singularity, arising from the on-shellness of l^2 when $l_T^2 = 0$. This gives a logarithmically singular integral dl_T^2/l_T^2. This happens when $l_T = 0$ and both l (the quark momentum) and $(p-l)$ (the gluon momentum) are on-shell and have positive + components, and therefore is of collinear origin. So the collinear divergence is a pinch singularity. Note that collinear divergence only requires two lines (the quark and the gluon propagators) to be collinear and on-shell, in contrast to the soft divergence where a soft gluon connects two on-shell quark lines so three lines are on-shell.

In the full DIS diagram (before factorization), the Wilson line here is replaced by a quark line with momentum $l + q$, whose virtuality is

$$(l+q)^2 = 2(l^+ + q^+)(l^- + q^-) - l_T^2 + (i\varepsilon) = -2(1-\alpha)p^+(l^- + q^-) - l_T^2 + \varepsilon \tag{164}$$

where we add the $i\varepsilon$ in its propagator to indicate the pole position. It gives the l^- pole on the same side as $(p-l)^2$, so still only $0 < \alpha < 1$ gives non-zero l^- integration. Due to the existence of the hard scale $q^- \sim Q$, the pole is to the left of the $(p-l)^2$ pole by distance q^-, so only l^2 and $(p-l)^2$ can be pinched for the l^- integral. In the Breit frame,

$$p = (p^+, 0, 0_T), \;\; p' = p + q = (0, p^+, 0_T), \;\; q = (-p^+, p^+, 0_T), Q^2 = -q^2 = 2(p^+)^2.$$

So, when picking the l^2 pole, we have

$$(l+q)^2 = -2(1-\alpha)p^+ \left(\frac{l_T^2}{2\alpha p^+} + p^+ \right) - l_T^2 = -(1-\alpha)Q^2 - \frac{1}{\alpha} l_T^2 \tag{165}$$

After performing the integration of l^- by picking the l^2 pole, we have an integral of α from 0 to 1. We notice that l_T^2 cuts off the singularity when $\alpha \to 1$. If we neglect l_T^2, we would a divergent result from $\int_0^1 d\alpha/(1-\alpha)$.

When keeping the l_T^2, the divergence at $\alpha \to 1$ turns to be a $\ln(l_T^2/Q^2)$. Together with the $\int dl_T^2/l_T^2$ from the pinched integration of l^-, we would get a double log of Q^2. If we use a lower cutoff Λ^2 for l_T, the leading singular structure is like $\ln^2(Q^2/m^2)$. This is singular at $m \to 0$, i.e., $l_T \to 0$. Note that one of the logs comes from the $l_T \to 0$ for a fixed $\alpha \neq 1$, which is collinear singularity; the other comes from α integration when $\alpha \to 1$ while taking $l_T \to 0$, or $(p-l) \to 0$, i.e., the gluon is soft. So the double-log singularity comes from the region where the gluon is *both soft and collinear.*

However, when getting to the l integration in the PDF, we found an unregulated divergent integral of α as $\alpha \to 1$. This is due to the Wilson line propagator $1/(l^+ - p^+)$. It arises from the eikonal approximation that we only keep l^+ in the other quark propagator in the hard process, which allows to detach the gluon from the hard part onto the gauge link. That is, we have thrown away the l_T^2 in Eq. (165) in the first place. However, after doing this the l^+ integration extends to infinity, and especially including the region where l_T cannot be ignored, which makes $l^+ \to p^+$ an unregulated singularity. At the l^- pole that we deform the contour to encircle, we have, for the quark and gluon momenta

$$q: \quad l = \left(\alpha p^+, \; -\frac{l_T^2}{2(1-\alpha)p^+}, \; \boldsymbol{l}_T \right) \tag{166}$$

$$g: \quad (p-l) = \left((1-\alpha)p^+, \; \frac{l_T^2}{2(1-\alpha)p^+}, \; \boldsymbol{l}_T \right) \tag{167}$$

For fixed l_T, as $\alpha \to 1$, the $+$ component of the gluon momentum goes to $0+$, but its $-$ component goes to $+\infty$, so it becomes very collinear to the Wilson line along n direction, having rapidity $y = 2^{-1}\ln[(p-l)^+/(p-l)^-] \to -\infty$. So we call this divergence *rapidity divergence*. The quark becomes far off-shell here. There will also be rapidity divergence from the real gluon emission diagram (b) in Fig. 10, and they will cancel exactly.

Combining Eq. (154) and (163) gives

$$f^{(a+a^\dagger)}_{(0)\,q/q}(x) = -\delta(1-x)\frac{g^2 C_F}{4\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\int_0^1 \frac{\alpha\, d\alpha}{1-\alpha}\int_0^\infty \frac{dl_T^2}{l_T^2}\left(l_T^2\right)^{-2\varepsilon} \tag{168}$$

where "h.c." became a factor 2 because the result of each diagram is real.

2. *Real gluon radiation, part one: (b)*

Diagram (b) in Fig. 10 represents a real gluon radiation. In the DIS factorization context, it comes from the interference between the real gluon emissions of the initial state quark and the final state quark and momentum region where the gluon is collinear to the initial state quark. In this case, the gluon is factorized into the Wilson line, as captured by this diagram. Following the Feynman rules, we have [13]

$$\begin{aligned} f^{(b+b^\dagger)}_{(0)\,q/q}(x) =& \int \frac{d^dk}{(2\pi)^d}\,\delta(k^+ - xp^+)\,(-g^{\mu\nu})\,(2\pi)\delta\left((p-k)^2\right)\,\theta(p^+-k^+)\times \\ &\times \mathrm{Tr}\left[\frac{\gamma^+}{2}\frac{i\not{k}}{k^2+i\varepsilon}\left(-ig\mu^\varepsilon\gamma_\nu T^a_{ji}\right)\frac{\not{p}}{2}\frac{-i}{n\cdot(p-k)-i\varepsilon}\left(ig\mu^\varepsilon n_\mu T^a_{ij}\right)\right] + \text{h.c.} \\ =& -g^2\mu^{2\varepsilon}C_F\int \frac{dk^- d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^d}(2\pi)\delta\left((p-k)^2\right)\frac{\mathrm{Tr}\left[\frac{\gamma^+}{2}\not{k}\gamma^+\frac{\not{p}}{2}\right]}{(p^+-k^+)\,(k^2+i\varepsilon)} + \text{h.c.} \end{aligned} \tag{169}$$

Using Eq. (160) allows to directly compare to Eq. (154)

$$f^{(a+a^\dagger)}_{(0)\,q/q}(x) = g^2\mu^{2\varepsilon}C_F\delta(p^+-xp^+)\int \frac{d^dl}{(2\pi)^d}(2\pi)\delta\left((p-l)^2\right)\frac{\mathrm{Tr}\left[\frac{\gamma^+}{2}\not{l}\gamma^+\frac{\not{p}}{2}\right]}{(p^+-l^+)\,(l^2+i\varepsilon)} + \text{h.c.} \tag{170}$$

We see that the integrations of $k^-, \boldsymbol{k}_T$ and $l^-, \boldsymbol{l}_T$ are the same. The only differences are

- k^+ is replaced by l^+ and is integrated, while both k^+ and l^+ are confined in $(0,1)$.
- There is a sign difference, because the momenta of the gauge links in (a) and (b) are opposite.

[13] From now on, we will omit the $\theta(1-x)$ function, keeping in mind that our calculation is valid only for $x > 0$ and the PDFs only have support at $x < 1$. This is enough for our purpose. One can get the PDFs at $x < 0$ using Eq. (38).

If there is no $\delta(k^+ - xp^+)$ function, both results would be exactly the same, except for a $-\delta(p^+ - xp^+)$ factor. $\delta(k^+ - xp^+)$ picks the value of the k^+ integrand at $k^+ = xp^+$. Since the ultimate result of Eq. (170) is an integration form of α (l^+), by picking its value at $l^+ = xp^+$ and removing the $-\delta(p^+ - xp^+)$ factor in Eq. (168), we can directly write down the result for Eq. (169):

$$f_{(0)\,q/q}^{(b+b^\dagger)}(x) = \frac{g^2 C_F}{4\pi^2} \frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)} \frac{x}{1-x} \int_0^\infty \frac{dk_T^2}{k_T^2} \left(k_T^2\right)^{-2\varepsilon} \tag{171}$$

This has the same rapidity divergence as $x \to 1$, but here x is not integrated. This will cancel the rapidity in $f_{(0)\,q/q}^{(a+a^\dagger)}(x)$, due to the oppsite signs. If we integrate over x from 0 to 1, these two diagrams would cancel exactly.

That the rapidity divergence must cancel between the virtual and read diagrams is because rapidity divergence is of infrared origin. When α or x go close to 1, the + momentum of the gluon line goes to 0, so does the gauge link propgator. This can be shown more clearly in light-cone gauge where there is no gauge link and the gauge link propagator turns into the special propagator $1/(n \cdot k)$ in the gluon propagator. So rapidity divergence occurs when this propagator goes soft. Since we do not observe those soft particles, they cancel between real and virtual diagrams, as a result of unitarity. This is also reflected in the fact that the two Wilson lines on the left and right of the cut line can be connected together into one link. If we observe the k_T the final states, the two Wilson lines can not be connected, and the soft singularities will not be canceled. In that case, they will become a Sudakov soft factor.

There is also collinear singularity at the limit when $k_T \to 0$. For the virtual diagram (a), we have demonstrated that the collinear divergence comes from the region where the quark and gluon lines are pinched as $l_T \to 0$, forcing the l^- contour to be close to the poles and leading to $1/l_T^2$ singularity. Here the gluon is a real particle in the final state, and l^- is fixed by the on-shell condition, for given k_T and k^+. We shall not talk about whether the contour can be deformed. But we do see that as k_T is small, the k^- value selected by the on-shell condition (the $\delta\left((p-k)^2\right)$ function) forces the virtuality of the quark line to be small:

$$k^2 = 2k^+ \left(\frac{-k_T^2}{2(p^+ - k^+)}\right) - k_T^2 = -\frac{p^+}{p^+ - k^+} k_T^2 = -\frac{1}{1-x} k_T^2 \to 0 \tag{172}$$

So as $k_T \to 0$, the (virtual) quark line also becomes on-shell. We also call such configuration a pinch by extending the concept. A pinch corresponds to a region where one or more propagators are forced to be close to mass shell. In this case, their propagators blow up and give a large and possibly divergent contribution to the integral.

Finally, both (a) and (b) contain UV divergences as l_T or $k_T \to \infty$. As explained before, this is an artifact due to factorization where we disentangle l_T or k_T from the hard process and *extend their integration range to infinity*. The UV divergence will be removed by the counterterms.

3. *Cancelation of rapidity divergence: the plus distribution*

When we sum over diagram (a) and (b), we get

$$\begin{aligned} f_{(0)\,q/q}^{(a+b)}(x) &= \frac{g^2 C_F}{4\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left(\frac{x}{1-x} - \delta(1-x)\int_0^1 \frac{\alpha\, d\alpha}{1-\alpha}\right)\int_0^\infty \frac{dl_T^2}{l_T^2}\left(l_T^2\right)^{-2\varepsilon} \\ &\equiv \frac{g^2 C_F}{4\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left(\frac{x}{1-x}\right)_+ \int_0^\infty \frac{dl_T^2}{l_T^2}\left(l_T^2\right)^{-2\varepsilon} \end{aligned} \tag{173}$$

where we formally defined a "plus" function of x. Generally, if a function $P(x)$ has singularity of x at $x = 1$ and $(1-x)P(x)$ is not singular, then we can define

$$\left[P(x)\right]_+ \equiv P(x) - \delta(1-x)\int_0^1 d\alpha P(\alpha) \tag{174}$$

The $\delta(1-x)$ formally takes away the singularity of $P(x)$ at $x = 1$. More strictly, this should be understood as an integration kernel, in the same sense of δ function. $[P(x)]_+$ is defined such that

$$\begin{cases} \left[P(x)\right]_+ = P(x) \text{ when } 0 < x < 1 \\ \displaystyle\int_0^1 dx \left[P(x)\right]_+ f(x) \equiv \int_0^1 dx P(x)(f(x) - f(1)) \end{cases} \tag{175}$$

where $f(x)$ is a regular function. We can use either Eq. (175) or Eq. (174) to perform algebraic calculations of the + distributions. For example, the plus distribution in Eq. (180) is

$$\begin{aligned} \left(\frac{x}{1-x}\right)_+ &= \frac{x}{1-x} - \delta(1-x)\int_0^1 dz \frac{z}{1-z} \\ &= -1 + \frac{1}{1-x} - \delta(1-x)\int_0^1 dz\left(-1 + \frac{1}{1-z}\right) \\ &= \frac{1}{(1-x)_+} - 1 + \delta(1-x) \end{aligned} \tag{176}$$

while a function times a plus distribution is easier to use Eq. (175) to transform, e.g., $x/(1-x)_+$ can be transformed using

$$\int_0^1 dx \frac{x}{(1-x)_+} f(x) = \int_0^1 dx \frac{x f(x) - f(1)}{(1-x)_+} = \int_0^1 dx \frac{(x f(x) - x f(1)) + (x-1) f(1)}{1-x}$$
$$= \int_0^1 dx \frac{x\,(f(x) - f(1))}{1-x} - f(1) = \int_0^1 dx \left[\left(\frac{x}{1-x} \right)_+ - \delta(1-x) \right] f(x) \tag{177}$$

to

$$\frac{x}{(1-x)_+} = \left(\frac{x}{1-x} \right)_+ - \delta(1-x) \tag{178}$$

So we have

$$\left(\frac{x}{1-x} \right)_+ = \frac{x}{(1-x)_+} + \delta(1-x) = \frac{1}{(1-x)_+} - 1 + \delta(1-x) \tag{179}$$

Therefore,

$$f^{(a+b)}_{(0)\,q/q}(x) \equiv \frac{g^2 C_F}{4\pi^2} \frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)} \left[\frac{1}{(1-x)_+} - 1 + \delta(1-x) \right] \int_0^\infty \frac{dk_T^2}{k_T^2} \left(k_T^2\right)^{-2\varepsilon} \tag{180}$$

This is well defined (free of singularity for x) in the sense of integration, which is all about the perturbative PDF. We will use it to derive the splitting kernel and calculate the hard coefficients, both of which only have physical meanings as integration kernels.

4. Real gluon radiation, part two: (c)

Diagram (c) in Fig. 10 represents the other real gluon radiation. In the DIS factorization context, it comes from a real collinear gluon emission of the initial state quark. In this case, the gluon is factorized into the Wilson line, as captured by this diagram. Following the Feynman rules, we have

$$f^{(c)}_{(0)\,q/q}(x) = \int \frac{d^d k}{(2\pi)^d} \delta(k^+ - x p^+) \left(-g^{\mu\nu}\right) (2\pi) \delta\left((p-k)^2\right) \theta(p^+ - k^+) \times$$
$$\times \operatorname{Tr}\left[\frac{\gamma^+}{2} \frac{i\not{k}}{k^2 + i\varepsilon} \left(-ig\mu^\varepsilon \gamma_\mu T^a_{ji}\right) \frac{\not{p}}{2} \left(ig\mu^\varepsilon \gamma_\nu T^a_{ij}\right) \frac{-i\not{k}}{k^2 - i\varepsilon} \right]$$
$$= -g^2 \mu^{2\varepsilon} C_F \int \frac{dk^- d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^d} (2\pi) \delta\left((p-k)^2\right) \frac{\operatorname{Tr}\left[\frac{\gamma^+}{2} \not{k} \gamma^\mu \frac{\not{p}}{2} \gamma_\mu \not{k}\right]}{(k^2 + i\varepsilon)(k^2 - i\varepsilon)} \tag{181}$$

There is no new feature in this diagram. And we proceed by integrating out the k^- using the δ function

$$\delta\left((k-p)^2\right) = \frac{1}{2(1-x)p^+}\delta\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right) \tag{182}$$

which then gives

$$k^2 = 2(xp^+)\left(-\frac{k_T^2}{2(1-x)p^+}\right) - k_T^2 = -\frac{k_T^2}{1-x} \tag{183}$$

$$\mathrm{Tr}[\cdots] = (2-d)\mathrm{Tr}\left[\frac{\gamma^+}{2}\not{k}\frac{\not{p}}{2}\not{k}\right] = (2-d)\left[2k^+k\cdot p - p^+k^2\right] = -2(1-\varepsilon)p^+k_T^2 \tag{184}$$

So we have

$$\begin{aligned}
f_{(0)\,q/q}^{(c)}(x) &= -g^2\mu^{2\varepsilon}C_F\int\frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\frac{1}{2(1-x)p^+}\left(-\frac{1-x}{k_T^2}\right)^2\left(-2(1-\varepsilon)p^+k_T^2\right)\\
&= g^2\mu^{2\varepsilon}C_F(1-\varepsilon)(1-x)\int\frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\frac{1}{k_T^2}\\
&= \frac{g^2C_F}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}(1-\varepsilon)(1-x)\int\frac{dk_T^2}{k_T^2}(k_T^2)^{-2\varepsilon}
\end{aligned} \tag{185}$$

Still, we have collinear and UV divergences. The collinear divergence comes from pinched quark propagator as $k_T \to 0$, just as in diagram (b). Note that there are two quark propagators going on-shell in this case, but the numerator contains a factor of k_T^2, so the collinear divergence is still logarithmic.

5. *UV subtraction and splitting kernel: combining (a)(b)(c)*

There is one more diagram (g) which concerns the external leg corrections. It also contains UV and collinear divergences. But the UV divergences are subtracted by the QCD counterterms from Z_2, the wavefunction renormalization coefficient of quark field. So the diagram (g) together with its counterterm does not give new UV divergence, and thus does not affect the PDF renormalization. We will deal with it in the next subsection.

By combining diagram (a)(b)(c), we have, for the bare PDF

$$f_{(0)\,q/q}^{(a+b+c)}(x) = \frac{g^2C_F}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left[\frac{2}{(1-x)_+} - 2 + 2\delta(1-x) + (1-\varepsilon)(1-x)\right]\int_0^\infty\frac{dk_T^2}{k_T^2}\left(k_T^2\right)^{-2\varepsilon} \tag{186}$$

which is 0 in DR. The counterterm $(Z_{qq}Z_2)^{[1]}(x)$ in Eq. (46) subtracts the UV divergence, so

$$(Z_{qq}Z_2)^{[1]}(z) = \frac{g^2 C_F}{8\pi^2}\left[\frac{2}{(1-z)_+} - 2 + 2\delta(1-z) + 1 - z\right]\left(-\frac{S_\varepsilon}{\varepsilon}\right)$$
$$= \left(-\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2 C_F}{8\pi^2}\left[\frac{2}{(1-z)_+} - 1 - z + 2\delta(1-z)\right] \tag{187}$$

which gives the UV renormalized PDF

$$f_{q/q}^{(a+b+c)}(x) = \left(-\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2 C_F}{8\pi^2}\left[\frac{2}{(1-x)_+} - 1 - x + 2\delta(1-x)\right] \tag{188}$$

where we have removed the subscript '(0)' to indicate that this is the renormalized PDF. The $1/\varepsilon$ pole represents the collinear divergence.

To obtain $Z_{qq}^{[1]}(z)$, we also need $Z_2^{[1]}$ in Feynman gauge, which is calculated in the appendix and is

$$Z_2^{[1]} = -\left(\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2 C_F}{16\pi^2}, \tag{189}$$

So

$$Z_{qq}^{[1]}(z) = (Z_{qq}Z_2)^{[1]}(z) - \delta(1-z)Z_2^{[1]}$$
$$= \left(-\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2 C_F}{8\pi^2}\left[\frac{2}{(1-z)_+} - 1 - z + \frac{3}{2}\delta(1-z)\right] \tag{190}$$

and the splitting kernel

$$\boxed{P_{qq}^{[1]}(z) = \frac{g^2 C_F}{8\pi^2}\left[\frac{2}{(1-z)_+} - 1 - z + \frac{3}{2}\delta(1-z)\right]} \tag{191}$$

Note that the PDF in Eq. (188) is not the ultimate form, for which we still need the contribution from diagram (g), which we now turn to.

6. *External leg correction (g)*

For the external (on-shell) leg corrections in (g) and its Hermitian conjugate, issues may arise for the on-shell propagators following the loop. The correct treatment is given by LSZ reduction formula, which tells us that the full effects of external on-shell leg corrections are *a factor of the residue of the full propagator at its pole.* A full quark propagator is given by

$$
\begin{aligned}
i\Pi_2(\not{p}) &= \frac{i}{\not{p}-m+i\varepsilon} + \frac{i}{\not{p}-m+i\varepsilon}\, i\Sigma_2(\not{p})\, \frac{i}{\not{p}-m+i\varepsilon} + \cdots \\
&= \frac{i}{\not{p}-m+i\varepsilon}\left[1 - \Sigma_2(\not{p})\frac{1}{\not{p}-m+i\varepsilon} + \left(-\Sigma_2(\not{p})\frac{1}{\not{p}-m+i\varepsilon}\right)^2 + \cdots\right] \\
&= \frac{i}{\not{p}-m+i\varepsilon}\,\frac{1}{1+\Sigma_2(\not{p})\frac{1}{\not{p}-m+i\varepsilon}} \\
&= \frac{i}{\not{p}-m+\Sigma_2(\not{p})+i\varepsilon}
\end{aligned}
\tag{192}
$$

where m is the renormalized mass (e.g., $\overline{\text{MS}}$ mass), not necessarily the physical pole mass m_P. The residue of $\Pi_2(\not{p})$ at its pole (the pole mass m_P) is given by

$$
\begin{aligned}
\text{residue }[\Pi_2(\not{p})] &= \lim_{\not{p}\to m_P}(\not{p}-m_P)\frac{1}{\not{p}-m+\Sigma_2(\not{p})+i\varepsilon} = \frac{1}{1+\Sigma_2'(\not{p}=m_P)} \\
&= 1-\Sigma_2'(\not{p}=m_P)+\cdots
\end{aligned}
\tag{193}
$$

where we only kept the terms relevant for the one-loop calculation. Taking a square root gives

$$
\sqrt{\text{residue }[\Pi_2(\not{p})]} = 1 - \frac{1}{2}\Sigma_2'(\not{p}=m_P)+\cdots \tag{194}
$$

At one-loop order, we only keep the first two terms and Σ_2 only contains the one-loop diagram and the leading-order counterterm, i.e.,

$$
\Sigma(\not{p}) = \Sigma^{(0)}(\not{p}) + Z_2^{[1]}\not{p} \tag{195}
$$

And then the square root of the residue

$$
\sqrt{\text{residue }[\Pi_2(\not{p})]} = 1 - \frac{1}{2}\left(\left.\frac{d\Sigma^{(0)}(\not{p})}{d\not{p}}\right|_{\not{p}=m_P} + Z_2^{[1]}\right) \tag{196}
$$

This is the factor to be multiplied to the external leg without any loop corrections (amputated leg). At 1-loop order, the effect is just to multiply the factor

$$
-\frac{1}{2}\left(\left.\frac{d\Sigma^{(0)}(\not{p})}{d\not{p}}\right|_{\not{p}=m_P} + Z_2^{[1]}\right) \tag{197}
$$

to the leg in the tree graph.

The $\Sigma^{(0)}(\not{p})$ and $Z_2^{[1]}$ in the massless case and $\overline{\text{MS}}$ scheme are provided in Appendix A 1 a, cf. Eq. (A2) and (A5). We can see that after taking a derivative of $\Sigma^{(0)}(\not{p})$ with respect to $\not{p}$ and then taking $\not{p}$ to the pole mass, which is 0 in massless case, we end up with scaleless integrals, which are 0 in DR. It also means that the bare $\Sigma^{(0)}$ contains both UV and collinear divergences. After subtracting UV divergence with $Z_2^{[1]}$, we are only left with collinear divergences. So the only effect of the external leg correction is to multiply the tree diagram by a factor

$$-\frac{1}{2}Z_2^{[1]} \tag{198}$$

which should be understood as the collinear divergence in the one-loop external leg correction.

So, diagram (g) and its Hermitian conjugate give

$$f_{q/q}^{(g+g^\dagger)}(x) = -\frac{1}{2}Z_2^{[1]} f_{q/q}^{[0]}(x) + \text{h.c.} = -Z_2^{[1]}\delta(1-x) = \frac{g^2\, C_F}{16\,\pi^2}\,\frac{S_\varepsilon}{\varepsilon}\delta(1-x) \tag{199}$$

Note that we did not add the subscript '(0)' because this is already renormalized.

7. *Final result, combining (a)(b)(c) with (g)*

Combing Eq. (188) and (199) gives the complete result for renormalized quark PDF in a massless quark target at 1-loop level

$$\boxed{f_{q/q}^{[1]}(x) = \left(-\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2 C_F}{8\pi^2}\left[\frac{2}{(1-x)_+} - 1 - x + \frac{3}{2}\delta(1-x)\right]} \tag{200}$$

D. NLO: Gluon in quark

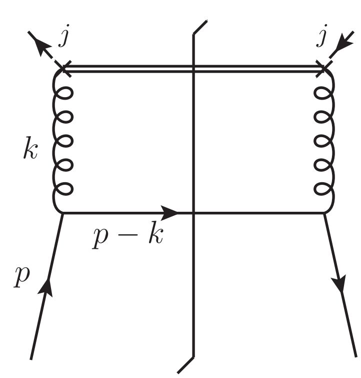


FIG. 14: NLO cut diagrams for gluon PDF in a quark target.

Now we turn to the calculation of gluon PDF. Contrary to quark PDF case, gluon PDF does not enter the NLO calculation of DIS, so we will not compare with the full DIS diagram before factorization, but the idea of how we end up with the factorized PDF diagrams is very similar. It can be understood by imagining a new current that interacts directly with gluons. Then we have the exact analogy to the quark PDF case.

For gluon PDF, the gauge link is in adjoint representation, and the gauge-link vertex should be changed accordingly. The gluon PDF in a quark target is easier since there is only one diagram, shown in Fig. 14, where we have used the same Lorentz index j for both gluon-link vertices to indicate an implicit sum over $j = 1, 2$, as required by the parton vertex in Eq. (53). Following the Feynman rules, we have

$$\begin{aligned}
f^{[1]}_{(0)g/q}(x) &= \frac{1}{xp^+}\int \frac{d^dk}{(2\pi)^d}\delta(k^+ - xp^+)\left(-i(k^+g^j_\mu - k^jn_\mu)\right)\frac{-i}{k^2+i\varepsilon}\left(i(k^+g^j_\nu - k^jn_\nu)\right)\frac{i}{k^2-i\varepsilon}\times \\
&\qquad \times \operatorname{Tr}\left[(2\pi)\delta\left((p-k)^2\right)(\not{p}-\not{k})\left(-ig\mu^\varepsilon\gamma^\mu T^a_{ji}\right)\frac{\not{p}}{2}\left(ig\mu^\varepsilon\gamma^\nu T^a_{ij}\right)\right] \\
&= \frac{1}{xp^+}g^2\mu^{2\varepsilon}C_F\int \frac{dk^-d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\delta\left((p-k)^2\right)\frac{1}{(k^2)^2}\times \\
&\qquad \times \left(k^+g^j_\mu - k^jn_\mu\right)\operatorname{Tr}\left[(\not{p}-\not{k})\gamma^\mu\frac{\not{p}}{2}\gamma^\nu\right]\left(k^+g^j_\nu - k^jn_\nu\right)
\end{aligned} \tag{201}$$

where $1/xp^+$ is from the gluon parton vertex, $\left(-i(k^+g^j_\mu - k^jn_\mu)\right)$ and $\left(i(k^+g^j_\nu - k^jn_\nu)\right)$ are the vertex where the gluon connects with the gauge link, and the minus sign in the gluon propagator is due to the $-g_{\mu\nu}$ numerator. a and ij are the color indices for gluon and quark, respectively, and a, j are summed over, which gives the C_F factor.

We have seen the $\delta\left((p-k)^2\right)\dfrac{1}{(k^2)^2}$ structures several times, in Fig. 8 for quark in a gluon and Fig. 10(c) for quark in a quark. They have the same *pinch structures*, and we can directly copy the results

$$\begin{aligned}
\delta\left((p-k)^2\right)\frac{1}{(k^2)^2} &= \delta\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right)\frac{1}{2(1-x)p^+}\left(\frac{1-x}{k_T^2}\right)^2 \\
&= \delta\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right)\frac{1-x}{2p^+}\frac{1}{(k_T^2)^2}
\end{aligned} \tag{202}$$

This naturally give a $1/k_T^4$ singularity. However, the numerator will give an extra k_T^2 factor so that the divergence is still logarithmic, as we will verify now.

The two gluon-link vertices give four terms, which we calculate one by one now. The first term is

$$
\begin{aligned}
&k^+ g^j_\mu \mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^\mu\,\frac{\not{p}}{2}\gamma^\nu\right] k^+ g^j_\nu = (k^+)^2\mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^j\,\frac{\not{p}}{2}\gamma^j\right] = -(k^+)^2\mathrm{Tr}\left[(\not{p}-\not{k})\,\frac{\not{p}}{2}\,\gamma^j\gamma^j\right]\\
=&(2-2\varepsilon)(k^+)^2\mathrm{Tr}\left[(\not{p}-\not{k})\,\frac{\not{p}}{2}\right] = (2-2\varepsilon)(k^+)^2(-2p\cdot k) = 2(1-\varepsilon)\,\frac{x^2}{1-x}\,(p^+)^2\,k_T^2 \qquad (203)
\end{aligned}
$$

where we have used the fact that $\not{p} = p^+\gamma^-$ anticommutes with γ^j and $\gamma^j\gamma^j = -(2-2\varepsilon)$. The second term is

$$
\begin{aligned}
&-k^j n_\mu \mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^\mu\,\frac{\not{p}}{2}\gamma^\nu\right] k^+ g^j_\nu = -k^j k^+ \mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^+\,\frac{\not{p}}{2}\gamma^j\right]\\
=&-2k^j k^+\left[(p-k)^j p^+\right] = 2x(p^+)^2 k_T^2 \qquad (204)
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&k^+ g^j_\mu \mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^\mu\,\frac{\not{p}}{2}\gamma^\nu\right](-k^j n_\nu) = -k^j k^+ \mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^j\,\frac{\not{p}}{2}\gamma^+\right]\\
=&-2k^j k^+\left[(p-k)^j p^+\right] = 2x(p^+)^2 k_T^2 \qquad (205)
\end{aligned}
$$

And the fourth term is

$$
\begin{aligned}
&(-k^j n_\mu)\mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^\mu\,\frac{\not{p}}{2}\gamma^\nu\right](-k^j n_\nu) = k_T^2\mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^+\,\frac{\not{p}}{2}\gamma^+\right]\\
=&2k_T^2\left[2(p-k)^+p^+\right] = 4(1-x)(p^+)^2 k_T^2 \qquad (206)
\end{aligned}
$$

So the numerator is

$$
\begin{aligned}
&(k^+ g^j_\mu - k^j n_\mu)\ \mathrm{Tr}\left[(\not{p}-\not{k})\,\gamma^\mu\,\frac{\not{p}}{2}\gamma^\nu\right](k^+ g^j_\nu - k^j n_\nu)\\
=&2(p^+)^2k_T^2\left[(1-\varepsilon)\frac{x^2}{1-x}+2x+2(1-x)\right] = 2(p^+)^2k_T^2\left[(1-\varepsilon)\frac{x^2}{1-x}+2\right] \qquad (207)
\end{aligned}
$$

We see that each term is proportional to k_T^2.

Plugging Eq. (202) and (207) in Eq. (201), we then have

$$
\begin{aligned}
f^{[1]}_{(0)g/q}(x) &= \frac{1}{xp^+}g^2\mu^{2\varepsilon}C_F\int\frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\frac{1-x}{2p^+}\frac{1}{(k_T^2)^2}\cdot 2(p^+)^2k_T^2\left[(1-\varepsilon)\frac{x^2}{1-x}+2\right]\\
&= \frac{g^2\,C_F}{8\,\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left[\frac{1+(1-x)^2}{x}-\varepsilon x\right]\int\frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon} \qquad (208)
\end{aligned}
$$

The UV pole is canceled by $Z^{[1]}_{gq}(x)$, so we have

$$
Z^{[1]}_{gq}(z) = -\left(\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2\,C_F}{8\,\pi^2}\left[\frac{1+(1-z)^2}{z}\right] \qquad (209)
$$

And then we have the renormalized gluon PDF in a quark target

$$\boxed{f_{g/q}^{[1]}(x) = -\left(\frac{S_\varepsilon}{\varepsilon}\right)\frac{g^2\,C_F}{8\,\pi^2}\left[\frac{1+(1-z)^2}{z}\right]} \tag{210}$$

and the splitting kernel

$$\boxed{P_{gq}^{[1]}(z) = \frac{g^2\,C_F}{8\,\pi^2}\left[\frac{1+(1-z)^2}{z}\right]} \tag{211}$$

E. NLO: Gluon in gluon

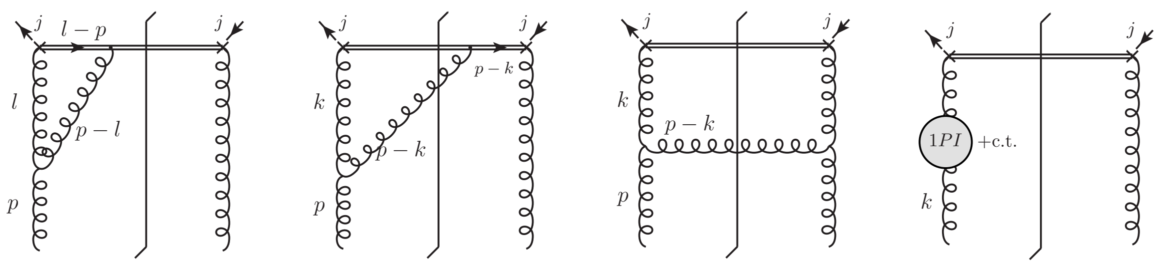


FIG. 15: The diagrams of one-loop gluon PDF in a gluon target. Diagrams (b)(c)(d) should be accompanied by their Hermitian conjugates. Three cut diagrams where both ends of a gluon propagator attach to the Wilson line vanish and are not shown.

The diagrams for gluon PDF in a gluon target are shown Fig. 15, where diagrams (b)(c)(d) should be accompanied by their Hermitian conjugates. Three cut diagrams where both ends of a gluon propagator attach to the Wilson line (like Fig. 11) vanish for the same reason and are not shown. The pinch structures and rapidity divergences (if any) follow exactly the diagrams for quark PDF in quark target, because they have the similar structures. These will not be analyzed in detail again. Although the pinch structures (propagator denominators) are the same, it is the numerators that make them have different results (and possibly different strengths of pinch singularities).

1. Virtual gluon radiation: (a)

Following the Feynman rules, diagram (a) and its Hermitian conjugate give

$$
\begin{aligned}
f^{(a+a^\dagger)}_{(0)\,g/g}(x) =& \frac{1}{xp^+}\delta(p^+ - xp^+)\int \frac{d^dl}{(2\pi)^d}\frac{i}{n\cdot(l-p)+i\varepsilon}\frac{-i}{l^2+i\varepsilon}\frac{-i}{(p-l)^2+i\varepsilon}\times \\
&\times\left[-i(p^+g^j_\rho - l^j n_\rho)\right]\left[i(p^+g^j_\nu - p^j n_\nu)\right](-g\mu^\varepsilon\, n_\lambda\, f_{cab})\,\varepsilon_\mu(p)\varepsilon^{\nu *}(p)\times \\
&\times(-g\mu^\varepsilon f_{abc})\left[(p+l)^\lambda g^{\mu\rho} + (p-2l)^\mu g^{\rho\lambda} + (l-2p)^\rho g^{\lambda\mu}\right] + \text{h.c.} \\
=& -ig^2\mu^{2\varepsilon}C_A\,\delta(1-x)\,\frac{1}{(p^+)^2}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^+-p^+}\frac{1}{l^2+i\varepsilon}\frac{1}{(p-l)^2+i\varepsilon} \\
&\varepsilon_\mu(p)(p^+g^j_\rho - l^j n_\rho)\left[(p+l)^+g^{\mu\rho} + (p-2l)^\mu n^\rho + (l-2p)^\rho n^\mu\right](p^+g^j_\nu - p^j n_\nu)\varepsilon^{\nu *}(p) + \text{h.c.}
\end{aligned}
\tag{212}
$$

where we have identified $k = p$ and used the $\delta(1-x)$ function to set $x = 1$ in the parton vertex. a, b, c are the gluon color factors and b, c are summed over to give C_A

$$
\sum_{b,c} f_{abc}f_{abc} = C_A = N_c. \tag{213}
$$

The propagators are the same as Eq. (154) for the diagram (a) in Fig. 10. So when the integral of l^- is only non-zero when $0 < l^+ < p^+$, and then l^2 and $(p-l)^2$ give l^- poles on the lower and upper half planes respectively. Picking the pole of $(p-l)^2$ gives, from Eq. (160) and (161),

$$
\begin{aligned}
\frac{1}{l^2+i\varepsilon}\,\frac{1}{(p-l)^2+i\varepsilon} &= -\frac{p^+-l^+}{p^+}\frac{1}{l_T^2}\frac{2\pi i}{-2(p^+-l^+)}\delta\left(l^- + \frac{l_T^2}{2(p^+-l^+)}\right) \\
&= \frac{i\pi}{p^+}\,\frac{1}{l_T^2}\,\delta\left(l^- + \frac{l_T^2}{2(p^+-l^+)}\right).
\end{aligned}
\tag{214}
$$

The numerator is independent of l^- and can be calculated directly

$$
\begin{aligned}
&\varepsilon_\mu(p)(p^+g^j_\rho - l^j n_\rho)\left[(p+l)^+g^{\mu\rho} + (p-2l)^\mu n^\rho + (l-2p)^\rho n^\mu\right](p^+g^j_\nu - p^j n_\nu)\varepsilon^{\nu *}(p) \\
=&(p^+g^j_\rho - l^j n_\rho)\left[(p+l)^+\varepsilon^\rho(p) - 2l\cdot\varepsilon(p)\,n^\rho\right]\left(p^+\,\varepsilon^{j*}(p)\right) \\
=&\left[p^+(p+l)^+\varepsilon^j(p)\right]\left(p^+\,\varepsilon^{j*}(p)\right) \\
=&(p^+)^2(p^+ + l^+)
\end{aligned}
\tag{215}
$$

where we used $0 = p\cdot\varepsilon(p) = n\cdot\varepsilon(p) = n^2 = n^j$ and $\sum_j \varepsilon^j(p)\varepsilon^{j*}(p) = 1$. Then we have

$$
\begin{aligned}
f^{(a+a^\dagger)}_{(0)\,g/g}(x) &= 2\pi g^2\mu^{2\varepsilon}C_A\,\delta(1-x)\,\frac{1}{(p^+)^3}\int_0^{p^+}dl^+\int\frac{d^{2-2\varepsilon}\boldsymbol{l}_T}{(2\pi)^d}\frac{1}{l^+-p^+}\,\frac{1}{l_T^2}(p^+)^2(p^++l^+) \\
&= -\,\delta(1-x)\,\frac{g^2C_A}{8\pi^2}\,\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\int_0^1 d\alpha\frac{1+\alpha}{1-\alpha}\int_0^\infty\frac{dl_T^2}{l_T^2}(l_T^2)^{-\varepsilon}
\end{aligned}
\tag{216}
$$

where the unregulated divergence at $\alpha \to 1$ reflects the rapidity divergence. As before, l_T integral contains logarithmic collinear divergence, arising from pinched l^2 and $(p-l)^2$ propagators, and logarithmic UV divergence as a result of extending l_T integration to infinity.

2. *Real gluon radiation, part one: (b)*

Diagram (b) together with its Hermitian conjugate gives

$$\begin{aligned} f^{(b+b^\dagger)}_{(0)\,g/g}(x) &= \frac{1}{xp^+}\int \frac{d^dk}{(2\pi)^d}\delta(k^+-xp^+)\frac{-i}{n\cdot(p-k)-i\varepsilon}\,\frac{-i}{k^2+i\varepsilon}\,(-2\pi)\delta\left((p-k)^2\right)\times \\ &\quad\times\left[-i(k^+g^j_\rho-k^jn_\rho)\right]\left[i(k^+g^j_\nu-p^jn_\nu)\right](+g\mu^\varepsilon\,n_\lambda\,f_{cab})\,\varepsilon_\mu(p)\varepsilon^{\nu*}(p)\times \\ &\quad\times(-g\mu^\varepsilon f_{abc})\left[(p+k)^\lambda g^{\mu\rho}+(p-2k)^\mu g^{\rho\lambda}+(k-2p)^\rho g^{\lambda\mu}\right]+\text{h.c.} \\ &= -g^2\mu^{2\varepsilon}C_A\frac{1}{xp^+}\int\frac{dk^-d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\,\frac{1}{p^+-k^+}\,\frac{1}{k^2+i\varepsilon}\,\delta\left((p-k)^2\right) \\ &\quad \varepsilon_\mu(p)(k^+g^j_\rho-k^jn_\rho)\left[(p+k)^+g^{\mu\rho}+(p-2k)^\mu n^\rho+(k-2p)^\rho n^\mu\right](k^+g^j_\nu-p^jn_\nu)\varepsilon^{\nu*}(p)+\text{h.c.} \end{aligned} \tag{217}$$

Note the '$-$' sign caused by the f^{abc} to the right of the cut line, which is highlighted by red font, as given by the first rule on p. 24. [14] Here we can not directly write down the result based on $f^{(a+a^\dagger)}_{(0)\,g/g}$, because although these two expressions are very similar, with k playing the role of l in $f^{(b+b^\dagger)}_{(0)\,g/g}$, there are still two differences:

- The vertices connecting gluon parton lines to the gauge link contain $k^+=xp^+$ in (b), while in (a) it is simply p^+;
- The parton vertex contains an extra factor of $1/x$ in (b), which is set to 1 by the $\delta(1-x)$ function in (a).

So unlike the quark PDF case where the result of the real emission is just the integrand of the virtual one, here we had better be careful to calculate explicitly. The δ function sets a pinch to the $1/k^2$ propagator when k_T is small:

$$\begin{aligned} \frac{1}{k^2+i\varepsilon}\,\delta\left((p-k)^2\right) &= \left(-\frac{p^+-k^+}{p^+}\,\frac{1}{k_T^2}\right)\frac{1}{2(p^+-k^+)}\,\delta\left(k^-+\frac{k_T^2}{2(p^+-k^+)}\right) \\ &= -\frac{1}{2p^+}\,\frac{1}{k_T^2}\,\delta\left(k^-+\frac{k_T^2}{2(p^+-k^+)}\right) \end{aligned} \tag{218}$$

[14] The effects caused by this sign are all fonted in red in the following.

The numerator is

$$\begin{aligned}
&\varepsilon_\mu(p)(k^+ g^j_\rho - k^j n_\rho)\left[(p+k)^+ g^{\mu\rho} + (p-2k)^\mu n^\rho + (k-2p)^\rho n^\mu\right](k^+ g^j_\nu - p^j n_\nu)\varepsilon^{\nu*}(p) \\
=&(k^+ g^j_\rho - k^j n_\rho)\left[(p+k)^+ \varepsilon^\rho(p) - 2k\cdot\varepsilon(p)\, n^\rho\right](k^+ \varepsilon^{j*}(p)) \\
=&(k^+)^2\,(p+k)^+ \varepsilon^j(p)\varepsilon^{j*}(p) = x^2(1+x)(p^+)^3 \qquad (219)
\end{aligned}$$

So, we have

$$\begin{aligned}
f^{(b+b^\dagger)}_{(0)\,g/g}(x) &= -g^2\mu^{2\varepsilon} C_A \frac{1}{xp^+}\int \frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}} \frac{1}{p^+ - k^+}\left(\frac{-1}{2p^+}\frac{1}{k_T^2}\right) x^2(1+x)(p^+)^3 + \text{h.c.} \\
&= +\frac{g^2 C_A}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\frac{x(1+x)}{1-x}\int \frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon} \qquad (220)
\end{aligned}$$

This also has rapidity divergence at $x \to 1$, but there is an extra factor x compared to the integrand of Eq. (216). The rapidity divergence will get canceled when combining (a) and (b).

3. Cancelation of rapidity divergence: combining (a) and (b)

Combing Eq. (216) and (220) gives

$$f^{(a+b)}_{(0)\,g/g}(x) = \frac{g^2 C_A}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left[\frac{x(1+x)}{1-x} - \delta(1-x)\int_0^1 d\alpha \frac{1+\alpha}{1-\alpha}\right]\int \frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon} \qquad (221)$$

The square bracket can be rewritten as a plus distribution:

$$\begin{aligned}
[\cdots] &= \frac{(x-1)(x+2)+2}{1-x} - \delta(1-x)\int_0^1 d\alpha \frac{-1+\alpha+2}{1-\alpha} \\
&= -x - 2 + \frac{2}{1-x} - \delta(1-x)\int_0^1 d\alpha\left(-1 + \frac{2}{1-\alpha}\right) \\
&= \frac{2}{(1-x)_+} - x - 2 + \delta(1-x) \qquad (222)
\end{aligned}$$

so

$$f^{(a+b)}_{(0)\,g/g}(x) = \frac{g^2 C_A}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left[\frac{2}{(1-x)_+} - x - 2 + \delta(1-x)\right]\int \frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon} \qquad (223)$$

which is well defined as a distribution of x, including the end point $x = 1$.

4. Real gluon radiation, part one: (c)

Diagram (c) contributes

$$f^{(c)}_{(0)\,g/g}(x) = \frac{1}{xp^+}\int \frac{d^dk}{(2\pi)^d}\delta(k^+ - xp^+)\frac{-i}{k^2+i\varepsilon}\,\frac{i}{k^2-i\varepsilon}\,(-2\pi)\delta\left((p-k)^2\right)\times$$
$$\times\left[-i(k^+g^j_\rho - k^jn_\rho)\right]\left[i(k^+g^j_\sigma - k^jn_\sigma)\right](-g\mu^\varepsilon f_{abc})(+g\mu^\varepsilon f_{abc})\varepsilon_\mu(p)\varepsilon^*_\nu(p)\times$$
$$\times\left[(p+k)^\lambda g^{\mu\rho} + (p-2k)^\mu g^{\rho\lambda} + (k-2p)^\rho g^{\lambda\mu}\right]\times$$
$$\times\left[-(p+k)^\lambda g^{\nu\sigma} + (2k-p)^\nu g^{\sigma\lambda} + (2p-k)^\sigma g^{\lambda\nu}\right]$$
$$= -g^2\mu^{2\varepsilon}C_A\frac{1}{xp^+}\int\frac{dk^-d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\frac{1}{(k^2)^2}\,\delta\left((p-k)^2\right)\times$$
$$\times\,\varepsilon_\mu(p)(k^+g^j_\rho - k^jn_\rho)\left[(p+k)^\lambda g^{\mu\rho} + (p-2k)^\mu g^{\rho\lambda} + (k-2p)^\rho g^{\lambda\mu}\right]\times$$
$$\times\,\varepsilon^*_\nu(p)(k^+g^j_\sigma - k^jn_\sigma)\left[(p+k)^\lambda g^{\nu\sigma} + (p-2k)^\nu g^{\sigma\lambda} + (k-2p)^\sigma g^{\lambda\nu}\right] \qquad (224)$$

Note again the '−' sign caused by the f^{abc} to the right of the cut line, which is highlighted by red font, as given by the first rule on p. 24. [15] We calculate each line one by one. The two propagators of k together with the δ function give, as in Eq. (202)

$$\delta\left((p-k)^2\right)\frac{1}{(k^2)^2} = \delta\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right)\frac{1-x}{2p^+}\,\frac{1}{(k_T^2)^2} \qquad (225)$$

The second line is denoted as $L^{j\lambda}$

$$L^{j\lambda} = \varepsilon_\mu(p)(k^+g^j_\rho - k^jn_\rho)\left[(p+k)^\lambda g^{\mu\rho} + (p-2k)^\mu g^{\rho\lambda} + (k-2p)^\rho g^{\lambda\mu}\right]$$
$$= (k^+g^j_\rho - k^jn_\rho)\left[(p+k)^\lambda\varepsilon^\rho(p) - 2k\cdot\varepsilon(p)g^{\rho\lambda} + (k-2p)^\rho\varepsilon^\lambda(p)\right]$$
$$= k^+\left[(p+k)^\lambda\varepsilon^j(p) - 2k\cdot\varepsilon(p)g^{j\lambda} + k^j\varepsilon^\lambda(p)\right] - k^j\left[-2k\cdot\varepsilon(p)n^\lambda + (k-2p)^+\varepsilon^\lambda(p)\right]$$
$$= 2\left(k^+g^{j\lambda} - k^jn^\lambda\right)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T) + k^+(k+p)^\lambda\varepsilon^j(p) + 2k^jp^+\varepsilon^\lambda(p) \qquad (226)$$

where in the last line we organized in terms of $\varepsilon(p)$, and we have used $k\cdot\varepsilon = -k^j\varepsilon^j = -\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T$. The third line $R^{j\lambda}$ can be obtained from $L^{j\lambda}$ by just changing ε to ε^*

$$R^{j\lambda} = 2\left(k^+g^{j\lambda} - k^jn^\lambda\right)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}^*_T) + k^+(k+p)^\lambda\varepsilon^{j*}(p) + 2k^jp^+\varepsilon^{\lambda*}(p) \qquad (227)$$

[15] The effects caused by this sign are all fonted in red in the following.

Then the product of $L^{j\lambda}$ and $R^{j\lambda}$ with j and λ contracted is [16]

$$
\begin{aligned}
L^{j\lambda}R^{j\lambda} = 4\left(k^+g^{j\lambda}-k^jn^\lambda\right)\left(k^+g^{j\lambda}-k^jn^\lambda\right)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T^*)+(k^+)^2(k+p)^2-4(p^+)^2k_T^2 \\
+\left[2k^+(k+p)^\lambda\left(k^+g^{j\lambda}-k^jn^\lambda\right)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T)\,\varepsilon^{j*}(p)+\text{c.c.}\right] \\
+\left[4p^+k^j\left(k^+g^{j\lambda}-k^jn^\lambda\right)\varepsilon^{\lambda*}(p)\,(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T)+\text{c.c.}\right] \\
+\left[-2k^+p^+(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T^*)+\text{c.c.}\right]
\end{aligned}
\tag{228}
$$

The 2nd-4th lines sum up to 0, so we have

$$
L^{j\lambda}R^{j\lambda} = 4(k^+)^2g^{jj}(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T^*)+(k^+)^2(k+p)^2-4(p^+)^2k_T^2 \tag{229}
$$

After integrating over k^-, the denorminator is spherically symmetric with $\boldsymbol{k}_T$, which allows us to do the replacement

$$
k^ik^j \to \frac{1}{2-2\varepsilon}k_T^2\delta^{ij} \tag{230}
$$

in the numerator. Then we have

$$
(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T)(\boldsymbol{k}_T\cdot\boldsymbol{\varepsilon}_T^*) \to \frac{1}{2-2\varepsilon}k_T^2\,\varepsilon^j(p)\varepsilon^{j*}(p) = \frac{1}{2-2\varepsilon}k_T^2 \tag{231}
$$

So,

$$
\begin{aligned}
L^{j\lambda}R^{j\lambda} &\to 4(k^+)^2\left(-(2-2\varepsilon)\right)\frac{1}{2-2\varepsilon}k_T^2+(k^+)^2(k+p)^2-4(p^+)^2k_T^2 \\
&= -4\,x^2\,(p^+)^2k_T^2-\frac{2x^2}{1-x}(p^+)^2k_T^2-4(p^+)^2k_T^2 \\
&= -2(p^+)^2k_T^2\left[2x^2+\frac{x^2}{1-x}+2\right]
\end{aligned}
\tag{232}
$$

Therefore, the diagram (c) is

$$
\begin{aligned}
f_{(0)\,g/g}^{(c)}(x) &= +g^2\mu^{2\varepsilon}C_A\frac{1}{xp^+}\int\frac{d^{2-2\varepsilon}\boldsymbol{k}_T}{(2\pi)^{d-1}}\frac{1-x}{2p^+}\frac{1}{(k_T^2)^2}\cdot 2(p^+)^2k_T^2\left[2x^2+\frac{x^2}{1-x}+2\right] \\
&= +\frac{g^2C_A}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left[2x(1-x)+x+\frac{2(1-x)}{x}\right]\int\frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon}
\end{aligned}
\tag{233}
$$

5. UV subtraction and splitting kernel: combining (a)(b)(c)

Combining (a)(b)(c) gives

$$
f_{(0)\,g/g}^{(a+b+c)}(x) = +\frac{g^2C_A}{8\pi^2}\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\left[2\left(\frac{1}{(1-x)_+}+x(1-x)-1+\frac{(1-x)}{x}\right)+\delta(1-x)\right]\int\frac{dk_T^2}{k_T^2}(k_T^2)^{-\varepsilon} \tag{234}
$$

[16] Here we carelessly write both λ as upper indices, in order to have a more symmetric and neat form. It should be understood as the usual Lorentz index contraction between upper and lower indices.

This is 0 in DR, but it contains UV and collinear divergences and is not 0 after renormalization. Diagram (d) does not give new UV divergence, so the UV pole of the gluon PDF is given by the above expression, which is subtracted by $(Z_{gg}Z_3)^{[1]}(x)$. So we have

$$(Z_{gg}Z_3)^{[1]}(x) = -\frac{S_\varepsilon}{\varepsilon}\frac{g^2 C_A}{8\pi^2}\left[2\left(\frac{1}{(1-x)_+} + x(1-x) - 1 + \frac{1-x}{x}\right) + \delta(1-x)\right] \tag{235}$$

And then the renormalized gluon PDF (including only (a)(b)(c), incomplete so far) is

$$f_{g/g}^{(a+b+c)}(x) = 0 + (Z_{gg}Z_3)^{[1]}(x) = -\frac{S_\varepsilon}{\varepsilon}\frac{g^2 C_A}{8\pi^2}\left[2\left(\frac{1}{(1-x)_+} + x(1-x) - 1 + \frac{1-x}{x}\right) + \delta(1-x)\right] \tag{236}$$

Z_{gg} is obtained by subtracting $Z_3^{[1]}$, the gluon field renormalization factor, which in Feynman gauge takes the form

$$Z_3^{[1]} = \frac{g^2}{8\pi^2}\frac{S_\varepsilon}{\varepsilon}\left[\frac{5}{6}C_A - \frac{2}{3}T_F n_f\right] \tag{237}$$

where n_f is the number of active quark flavors. So we have

$$\begin{aligned} Z_{gg}^{[1]}(z) &= (Z_{gg}Z_3)^{[1]}(z) - Z_3^{[1]}\delta(1-z) \\ &= -\frac{S_\varepsilon}{\varepsilon}\frac{g^2}{8\pi^2}\left[2C_A\left(\frac{1}{(1-z)_+} + z(1-z) - 1 + \frac{1-z}{z}\right) + \left(\frac{11}{6}C_A - \frac{2}{3}n_f T_F\right)\delta(1-z)\right] \end{aligned} \tag{238}$$

and the $g \to g$ splitting/evolution kernel

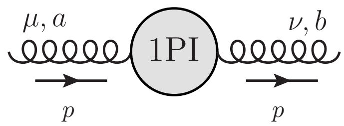
$$\begin{aligned} P_{gg}^{[1]}(z) &= \frac{g^2}{8\pi^2}\left[2C_A\left(\frac{1}{(1-z)_+} + z(1-z) - 1 + \frac{1-z}{z}\right) + \left(\frac{11}{6}C_A - \frac{2}{3}n_f T_F\right)\delta(1-z)\right] \\ &= \boxed{\frac{g^2}{8\pi^2}\left[2C_A\left(\frac{z}{(1-z)_+} + z(1-z) + \frac{1-z}{z}\right) + \left(\frac{11}{6}C_A - \frac{2}{3}n_f T_F\right)\delta(1-z)\right]} \end{aligned} \tag{239}$$

In the second line we used the algebra of plus distribution (also in Eq. (179)) to get the same form as Collins Eq. (9.24).

6. *External leg correction (d)*

Now we include the external leg correction (d). Similar to the quark case, the whole effect of the radiative correction on the external on-shell leg is to give a factor of the square root of the residue of the full (renormalized) propagator at its pole. We denote the one-particle-irreducible (1PI) two-point gluon function as

$$i\Pi^{\mu\nu}_{ab}(p) =$$

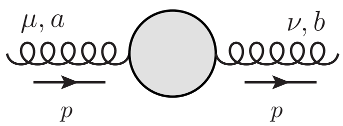


$$= \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\delta_{ab}\, i\Pi_2(p^2) \tag{240}$$

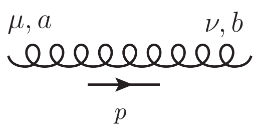
where the counterterms have been included and the tensor structure $(p^2 g^{\mu\nu} - p^\mu p^\nu)$ has been factored out by virtue of Ward identity. At $\mathcal{O}(g^2)$ order, $\Pi_2(p^2)$ has been calculated in Appendix A 2 a.

In the general R_ξ gauge, the full gluon propagator is

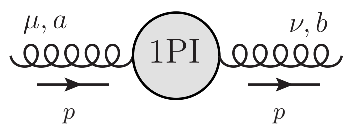
$$iG^{\mu\nu}_{ab}(p) =$$



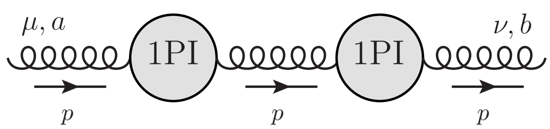
$$=$$



$$+$$



$$+$$



$$+\cdots$$

$$\begin{aligned}
&= \frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + \xi\frac{p^\mu p^\nu}{p^2}\right)\delta_{ab} \\
&\quad + \frac{-i}{p^2+i\varepsilon}\left(g^{\mu}_{\lambda} - \frac{p^\mu p_\lambda}{p^2} + \xi\frac{p^\mu p_\lambda}{p^2}\right)\delta_{ac}\cdot\left[i\Pi_2(p^2)\delta_{cd}(p^2 g^{\lambda}_{\rho} - p^\lambda p_\rho)\right]\times \\
&\qquad\qquad \times\frac{-i}{p^2+i\varepsilon}\left(g^{\rho\nu} - \frac{p^\rho p^\nu}{p^2} + \xi\frac{p^\rho p^\nu}{p^2}\right)\delta_{cb} + \cdots \\
&= \frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + \xi\frac{p^\mu p^\nu}{p^2}\right)\delta_{ab} + \frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi(p^2)\delta_{ab}\cdots \\
&= \frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta_{ab}\left[1+\Pi(p^2)+\Pi^2(p^2)+\cdots\right] + \frac{-i}{p^2+i\varepsilon}\cdot\xi\frac{p^\mu p^\nu}{p^2}\delta_{ab} \\
&= \frac{-i}{p^2\left[1-\Pi(p^2)\right]+i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta_{ab} + \frac{-i}{p^2+i\varepsilon}\cdot\xi\frac{p^\mu p^\nu}{p^2}\delta_{ab}
\end{aligned} \tag{241}$$

where we have separated the two tensor projection structures $g^{\mu\nu} - p^\mu p^\nu/p^2$ and $p^\mu p^\nu/p^2$, and used

$$\left(g^\mu_\lambda - \frac{p^\mu p_\lambda}{p^2}\right)\left(g^\lambda_\nu - \frac{p^\lambda p_\nu}{p^2}\right) = \left(g^\mu_\nu - \frac{p^\mu p_\nu}{p^2}\right) \tag{242}$$

$$\left(g^\mu_\lambda - \frac{p^\mu p_\lambda}{p^2}\right)\frac{p^\lambda p_\nu}{p^2} = 0 \tag{243}$$

$$\frac{p^\mu p_\lambda}{p^2}\frac{p^\lambda p_\nu}{p^2} = \frac{p^\mu p_\nu}{p^2} \tag{244}$$

The transverse structure of 1PI diagrams leave the ξ term unchanged, and keep the pole of the propagator at $p^2 = 0$. But it modifies to residue to be

$$\text{residue} = \frac{1}{1-\Pi(0)} = 1 + \Pi(0) + \cdots \tag{245}$$

Thus the one-loop correction of the gluon externel leg gives a factor

$$\sqrt{\text{residue}} = \frac{1}{\sqrt{1-\Pi(0)}} = 1 + \frac{1}{2}\Pi(0) + \mathcal{O}(g^4) \tag{246}$$

From Appendix A 2 a f, $\Pi(p^2)$ contains the bare term and the counterterm

$$\Pi(p^2) = \Pi_{(0)}(p^2) - Z_3^{[1]} \tag{247}$$

where $Z_3^{[1]}$ is given in Eq. (A36). When taking p^2 to 0, $\Pi_{(0)}(0)$ is a scaleless integral in DR and thus vanishes. And then we have

$$\Pi(0) = -Z_3^{[1]} \tag{248}$$

So the external diagram (d) contributes a factor $-2^{-1}Z_3^{[1]}$ to the bare gluon PDF. Together with its Hermitian conjugate, we have

$$f_{g/g}^{(g+g^\dagger)}(x) = -Z_3^{[1]}\delta(1-x) \tag{249}$$

7. Final result, combining (a)(b)(c) with (d)

Combing Eq. (236) and (249) gives the complete result for renormalized qluon PDF in a gluon target at 1-loop level

$$\boxed{f_{g/g}^{[1]}(x) = -\frac{S_\varepsilon}{\varepsilon}\frac{g^2}{8\pi^2}\left[2C_A\left(\frac{1}{(1-x)_+} + x(1-x) - 1 + \frac{1-x}{x}\right) + \left(\frac{11}{6}C_A - \frac{2}{3}n_f T_F\right)\delta(1-x)\right]} \tag{250}$$

Appendix A: Field Renormalization

1. Quark field renormalization: Z_2

a. *Feynman gauge*

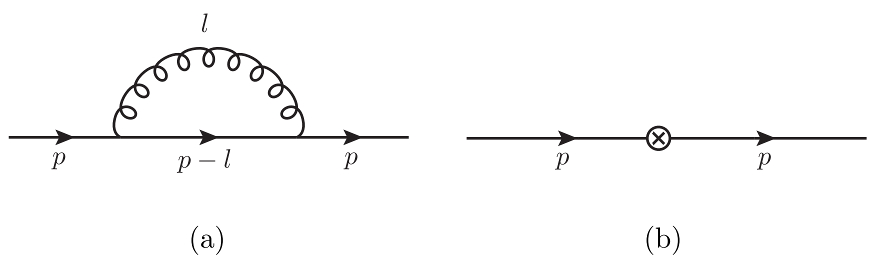


FIG. 16: Self energy diagrams for quark in QCD. (a) is the one-loop diagram. (b) is the counterterm interaction.

The quark field renormalization is related to diagrams shown in Fig. 16. In Feynman gauge, the diagram (a) gives

$$\begin{aligned} i\Sigma_2^{(0)} &= (-ig\mu^\varepsilon\gamma^\mu T^a_{ik})\int\frac{d^dl}{(2\pi)^d}\frac{i(\not p-\not l)}{(p-l)^2+i\varepsilon}\frac{-ig_{\mu\nu}}{l^2+i\varepsilon}(-ig\mu^\varepsilon\gamma^\nu T^a_{kj}) \\ &= -g^2\mu^{2\varepsilon}C_F\delta_{ij}\int\frac{d^dl}{(2\pi)^d}\frac{\gamma^\mu(\not p-\not l)\gamma_\mu}{(l^2+i\varepsilon)\left((p-l)^2+i\varepsilon\right)} \\ &= (d-2)g^2\mu^{2\varepsilon}C_F\delta_{ij}\int\frac{d^dl}{(2\pi)^d}\frac{\not p-\not l}{(l^2+i\varepsilon)\left((p-l)^2+i\varepsilon\right)} \end{aligned} \tag{A1}$$

where we have used massless partons. Using Feynman parametrization for the propagators gives

$$\begin{aligned} i\Sigma_2^{(0)} &= (d-2)g^2\mu^{2\varepsilon}C_F\delta_{ij}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{\not p-\not l}{\left[(l-xp)^2-x(x-1)p^2+i\varepsilon\right]^2} \\ &= 2(1-\varepsilon)g^2\mu^{2\varepsilon}C_F\delta_{ij}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{(1-x)\not p}{\left[l^2-x(x-1)p^2+i\varepsilon\right]^2} \end{aligned} \tag{A2}$$

This has UV divergence

$$\begin{aligned} \mathrm{UV}(i\Sigma_2^{(0)}) &= 2g^2C_F\delta_{ij}\int_0^1 dx(1-x)\not p\left[\frac{i}{(4\pi)^2}\frac{S_\varepsilon}{\varepsilon}\right] \\ &= \frac{i}{(4\pi)^2}\frac{S_\varepsilon}{\varepsilon}g^2C_F\delta_{ij}\not p \end{aligned} \tag{A3}$$

This is canceled by the counterterm diagram (b)

$$i(Z_2 - 1)^{[1]}\ \delta_{ij}\, \not{p} \tag{A4}$$

so we have

$$(Z_2 - 1)^{[1]} = Z_2^{[1]} = -\frac{g^2\, C_F}{16\,\pi^2}\,\frac{S_\varepsilon}{\varepsilon} \tag{A5}$$

where the superscript '[1]' denotes the perturbation order in terms of α_s.

2. Gluon field renormalization: Z_3

a. Feynman gauge

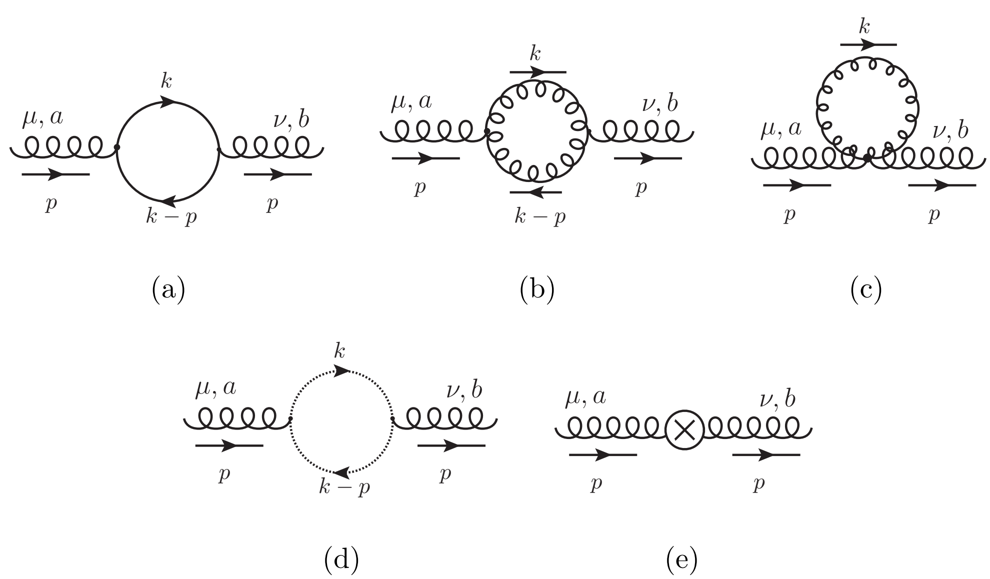


FIG. 17: Self energy diagrams for gluon in QCD. (a) is the quark loop diagram. (b) is gluon bubble diagram. (c) is the seagull diagram. (d) is the ghost loop diagram. (e) is the counterterm diagram.

Gluon self-energy diagrams at 1-loop level are shown in Fig. 17. The superficial degree of UV divergence is quadratic, which shows up as a pole at $d = 2$ in DR, but due to gauge invariance, quadratic divergences cancel and we only have UV pole at $d = 4$. The sum of them have the structure

$$i\Pi^{\mu\nu}_{ab}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\,\delta_{ab}\, i\Pi_2(p^2) \tag{A6}$$

holding for any value of d.

a. Fermion Loop. Diagram (a) contributes only log divergence and it alone satisfies the structure in Eq. (A6) because Ward identity holds for this diagram alone. It gives

$$
\begin{aligned}
i\Pi^{(a)\mu\nu}_{ab}(p) &= -n_f \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\left[\left(-ig\mu^\varepsilon T^a_{ij}\gamma^\mu\right)\frac{i(\not{k}-\not{p})}{(k-p)^2+i\varepsilon}\left(-ig\mu^\varepsilon T^b_{ji}\gamma^\nu\right)\frac{i\not{k}}{k^2+i\varepsilon}\right] \\
&= -g^2\mu^{2\varepsilon} n_f T_F\,\delta_{ab}\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p})\gamma^\nu\not{k}\right]}{\left((k-p)^2+i\varepsilon\right)\left(k^2+i\varepsilon\right)}
\end{aligned} \tag{A7}
$$

where we used massless quarks and n_f is the number of active quark flavors. i, j are the color indices of the quarks in the loop and are summed over, giving T_F. Performing Feynman parametrization to the denominators gives

$$
\frac{1}{\left((k-p)^2+i\varepsilon\right)\left(k^2+i\varepsilon\right)} = \int_0^1 dx \frac{1}{\left[(k-xp)^2 - x(x-1)p^2 + i\varepsilon\right]^2} \tag{A8}
$$

Then we shift $k \to k + xp$ in the integrand and keep only the even powers of k. The numerator becomes

$$
\begin{aligned}
\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p})\gamma^\nu\not{k}\right] &\to 4\left[2k^\mu k^\nu - k^2 g^{\mu\nu} + 2x(x-1)p^\mu p^\nu - x(x-1)p^2 g^{\mu\nu}\right] \\
&\to 4\left[\frac{2}{d}k^2\,g^{\mu\nu} - k^2 g^{\mu\nu} + 2x(x-1)\,p^\mu p^\nu - x(x-1)p^2\,g^{\mu\nu}\right] \\
&= 4\left[\left(\frac{2}{d}-1\right)\,k^2\,g^{\mu\nu} + 2x(x-1)\,p^\mu p^\nu - x(x-1)p^2\,g^{\mu\nu}\right]
\end{aligned} \tag{A9}
$$

where in the second line we replaced $k^\mu k^\nu$ by $\frac{1}{d}k^2 g^{\mu\nu}$ since they give the same result after k integration. We see that the $d = 2$ pole from the k^2 term is canceled by the coefficient $2/d - 1$ which is 0 at $d = 2$, so we only have a pole at $d = 4$, representing logarithmic divergence. Then we have

$$
i\Pi^{(a)\mu\nu}_{ab}(p) = -4g^2\mu^{2\varepsilon} n_f T_F\,\delta_{ab}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{\left(\frac{2}{d}-1\right)\,k^2\,g^{\mu\nu} + x(x-1)\,\left(2p^\mu p^\nu - p^2\,g^{\mu\nu}\right)}{\left[k^2 - x(x-1)p^2 + i\varepsilon\right]^2} \tag{A10}
$$

The k integration gives two parts

$$
\begin{aligned}
\int \frac{d^dk}{(2\pi)^d}\frac{\left(\frac{2}{d}-1\right)\,k^2}{\left[k^2 - x(x-1)p^2 + i\varepsilon\right]^2} &= -\left(\frac{2}{d}-1\right)\frac{d}{2}\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{\left[-x(1-x)p^2 - i\varepsilon\right]^{1-d/2}} \\
&= -\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\left[-x(1-x)p^2 - i\varepsilon\right]^{1-d/2}}
\end{aligned} \tag{A11}
$$

and

$$
\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2 - x(x-1)p^2 + i\varepsilon\right]^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\left[-x(1-x)p^2 - i\varepsilon\right]^{2-d/2}} \tag{A12}
$$

so the k integral is

$$\int dk[\cdots] = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{[-x(1-x)p^2 - i\varepsilon]^{2-d/2}} \left[x(1-x)p^2 g^{\mu\nu} + x(x-1)\left(2p^\mu p^\nu - p^2\, g^{\mu\nu}\right)\right]$$
$$= \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{[-x(1-x)p^2 - i\varepsilon]^{2-d/2}} \left[2x(1-x)\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\right] \tag{A13}$$

where the tensor structure $(p^2 g^{\mu\nu} - p^\mu p^\nu)$ appears before integrating over x. We then have

$$i\Pi^{(a)\mu\nu}_{ab}(p) = -\delta_{ab}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \frac{8ig^2}{(4\pi)^2} n_f T_F \Gamma(\varepsilon) \int_0^1 dx\, x(1-x) \left(\frac{4\pi\mu^2}{-x(1-x)p^2 - i\varepsilon}\right)^{\varepsilon}$$
$$= -\delta_{ab}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \frac{8ig^2}{(4\pi)^2} n_f T_F \int_0^1 dx\, x(1-x) \left[\frac{1}{\varepsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{-x(1-x)p^2 - i\varepsilon}\right)\right] \tag{A14}$$

The $1/\varepsilon$ is the UV pole arising from the loop momentum $k \to \infty$. There is also a singularity as $p^2 \to 0$, which is a collinear singularity. It arises from the configuration where quark and antiquark are collinear to the gluon line and propagate in the forward direction, corresponding to a pinch surface

$$k^\mu = z\, p^\mu, \quad (p-k)^\mu = (1-z)\, p^\mu, \quad 0 < z < 1 \tag{A15}$$

The collinear singularity and the UV pole add up to 0 when $p^2 = 0$. This can be seen from Eq. (A10), which becomes a scaleless integral in DR when we take $p^2 = 0$, so is 0. The UV pole is

$$\text{UV}\ \{i\Pi^{(a)\mu\nu}_{ab}(p)\} = -\delta_{ab}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \frac{ig^2}{(4\pi)^2} \frac{4}{3}\, n_f T_F\, \frac{S_\varepsilon}{\varepsilon} \tag{A16}$$

which will be canceled by the counterterm diagram (e). The renormalized diagram (a) is then

$$i\Pi^{(a,ren)\mu\nu}_{ab}(p) = -\delta_{ab}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \frac{ig^2}{(4\pi)^2} n_f T_F \left[\frac{4}{3} \ln\left(\frac{\mu^2}{-p^2 - i\varepsilon}\right) + \frac{20}{9}\right] \tag{A17}$$

We have kept $i\varepsilon$ with $-p^2$ to make it clear how to deal with negative $-p^2$ when we need to pick up the imaginary part.

b. *Gluon bubble.* Diagram (b) gives

$$\begin{aligned}
i\Pi_{ab}^{(b)\mu\nu}(p) &= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{-i}{(k-p)^2+i\varepsilon}\frac{-i}{k^2+i\varepsilon}\times \\
&\qquad \times(-g\mu^\varepsilon f_{acd})\left[(p+k)^\sigma g^{\mu\rho}+(p-2k)^\mu g^{\rho\sigma}+(k-2p)^\rho g^{\sigma\mu}\right]\times \\
&\qquad \times(-g\mu^\varepsilon f_{bcd})\left[-(p+k)^\sigma g^{\nu\rho}-(p-2k)^\nu g^{\rho\sigma}-(k-2p)^\rho g^{\sigma\nu}\right] \\
&= \frac{1}{2}g^2\mu^{2\varepsilon}C_A\,\delta_{ab}\int \frac{d^dk}{(2\pi)^d}\frac{N^{\mu\nu}}{((k-p)^2+i\varepsilon)\,(k^2+i\varepsilon)}\times \\
&\qquad \times\left[(p+k)^\sigma g^{\mu\rho}+(p-2k)^\mu g^{\rho\sigma}+(k-2p)^\rho g^{\sigma\mu}\right]\times \\
&\qquad \times\left[(p+k)^\sigma g^{\nu\rho}+(p-2k)^\nu g^{\rho\sigma}+(k-2p)^\rho g^{\sigma\nu}\right]
\end{aligned} \tag{A18}$$

where the numerator $N^{\mu\nu}$ is

$$\begin{aligned}
N^{\mu\nu} &= \left[(p+k)^\sigma g^{\mu\rho}+(p-2k)^\mu g^{\rho\sigma}+(k-2p)^\rho g^{\sigma\mu}\right]\cdot\left[(p+k)^\sigma g^{\nu\rho}+(p-2k)^\nu g^{\rho\sigma}+(k-2p)^\rho g^{\sigma\nu}\right] \\
&= (p+k)^2 g^{\mu\nu}+d\,(p-2k)^\mu(p-2k)^\nu+(k-2p)^2 g^{\mu\nu} \\
&\quad +\left[(p+k)^\mu(p-2k)^\nu+(p+k)^\nu(k-2p)^\mu+(p-2k)^\mu(k-2p)^\nu+(\mu\leftrightarrow\nu)\right]
\end{aligned} \tag{A19}$$

Using the Feynman parametrization formula in Eq. (A8), we replace k by $k+xp$, throw away the linear term of k, and obtain the numerator

$$\begin{aligned}
N^{\mu\nu} &\to 2(2d-3)k^\mu k^\nu+2k^2g^{\mu\nu}+\left[d(1-2x)^2-6\left(x^2-x+1\right)\right]p^\mu p^\nu+p^2g^{\mu\nu}\left(2x^2-2x+5\right) \\
&\to \frac{2}{d}(2d-3)k^2g^{\mu\nu}+2k^2g^{\mu\nu}+\left[d(1-2x)^2-6\left(x^2-x+1\right)\right]p^\mu p^\nu+p^2g^{\mu\nu}\left(2x^2-2x+5\right) \\
&= 6\left(1-\frac{1}{d}\right)k^2g^{\mu\nu}+\left[d(1-2x)^2-6\left(x^2-x+1\right)\right]p^\mu p^\nu+p^2g^{\mu\nu}\left(2x^2-2x+5\right) \\
&\equiv N_b^{\mu\nu}
\end{aligned} \tag{A20}$$

where in the second line we replaced $k^\mu k^\nu$ by $\frac{1}{d}k^2g^{\mu\nu}$. So diagram (b) gives

$$i\Pi_{ab}^{(b)\mu\nu}(p) = \frac{1}{2}g^2\mu^{2\varepsilon}C_A\,\delta_{ab}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{N_b^{\mu\nu}}{\left[k^2-x(x-1)p^2+i\varepsilon\right]^2} \tag{A21}$$

with the new numerator given by Eq. (A20). We notice that the $d=2$ pole is not canceled, so this diagram alone does not satisfy gauge invariance or Ward identity. Only the sum of (b)(d)(e) will do.

c. Four-point gluon bubble. Diagram (c) gives

$$i\Pi_{ab}^{(c)\mu\nu}(p) = \frac{1}{2}(-ig^2\mu^{2\varepsilon})\int \frac{d^dk}{(2\pi)^d}\frac{-i}{k^2+i\varepsilon}\times$$
$$\times\left[f_{acd}f_{bcd}\left(g^{\mu\nu}g^{\rho\rho}-g^{\mu\rho}g^{\rho\nu}\right)+f_{acd}f_{bcd}\left(g^{\mu\nu}g^{\rho\rho}-g^{\mu\rho}g^{\rho\nu}\right)\right]$$
$$= -g^2\mu^{2\varepsilon}C_A\delta_{ab}(d-1)g^{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\varepsilon} \tag{A22}$$

This is a scaleless integral in DR and formally vanishes. But it contains quadratic divergence, which shall cancel part of that in diagram (b). In DR, the quadratic divergence cancels against the logarithmic divergence. Although the final result will not be affected, it is convenient to separate the quadratic and log divergences and see the explicit cancelation of quadratic divergence. By multipling a factor $(k-p)^2/(k-p)^2$ and using Feynman parametrization (A8) we get

$$i\Pi_{ab}^{(c)\mu\nu}(p) = -g^2\mu^{2\varepsilon}C_A\delta_{ab}(d-1)g^{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\frac{(k-p)^2}{(k^2+i\varepsilon)\left((k-p)^2+i\varepsilon\right)}$$
$$= -g^2\mu^{2\varepsilon}C_A\delta_{ab}(d-1)g^{\mu\nu}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{k^2+(1-x)^2p^2}{\left[k^2-x(x-1)p^2+i\varepsilon\right]^2} \tag{A23}$$

where k^2 gives quadratic divergence and $(1-x)^2p^2$ gives logarithmic divergence. Despite the appearance, this integral still vanishes after integrating over k and x (for arbitrary d).

Note that the combination of (b) and (c) does not cancel the pole at $d=2$. This is a sign of a chain of violations: first, this means that gluon will acquire a mass through radiative corrections; second, gauge invariance is not retained; third, a longitudinal component of gluon field (a new degree of freedom) is generated, violating unitarity; fourth, Ward identity is not satisfied (as a result of gauge invariance breaking and longitudinal component). This means that the ghost bubble (d) is necessary to restore gauge invariance, unitarity, and Ward identity.

d. Ghost bubble. The ghost bubble diagram diagram (d) gives

$$i\Pi_{ab}^{(d)\mu\nu}(p) = -\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2+i\varepsilon}\frac{i}{(k-p)^2+i\varepsilon}\left(gf_{dac}k^\mu\right)\left(gf_{cbd}(k-p)^\nu\right)$$
$$= -g^2\mu^{2\varepsilon}C_A\delta_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu(k-p)^\nu}{(k^2+i\varepsilon)\left((k-p)^2+i\varepsilon\right)}$$
$$= -g^2\mu^{2\varepsilon}C_A\delta_{ab}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{\frac{1}{d}k^2g^{\mu\nu}+x(x-1)p^\mu p^\nu}{\left[k^2-x(x-1)p^2+i\varepsilon\right]^2} \tag{A24}$$

where the overall negative sign is from the ghost loop and in the third line we have performed Feynman parametrization, shifted the momentum k and replaced $k^\mu k^\nu$ by $\dfrac{1}{d}k^2g^{\mu\nu}$.

e. Combination of (a)(b)(c)(d). First we shall see the cancelation of quadratic divergence among the gluon loop and ghost loop. The k^2 coefficient in the sum of (b)(c)(d) is

$$\frac{1}{2}\cdot 6\left(1-\frac{1}{d}\right)-(d-1)-\frac{1}{d}=-\frac{(d-2)^2}{d} \tag{A25}$$

apart from the overall coefficient $g^2\mu^{2\varepsilon}C_A\delta_{ab}$. It indeed gives a zero at $d=2$ and cancels the pole at $d=2$. So we are left with only log divergence.

Summing over diagrams (b)(c)(d) gives

$$i\Pi_{ab}^{(b+c+d)\mu\nu}(p)=g^2\mu^{2\varepsilon}C_A\delta_{ab}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{N^{\mu\nu}}{\left[k^2-x(x-1)p^2+i\varepsilon\right]^2} \tag{A26}$$

with the new numerator $N^{\mu\nu}$ being

$$\begin{aligned}N^{\mu\nu}=&-\frac{(d-2)^2}{d}k^2g^{\mu\nu}+\frac{1}{2}p^2g^{\mu\nu}\left(2x^2-2x+5\right)-(d-1)(1-x)^2p^2g^{\mu\nu}\\&+\frac{1}{2}\left[d(1-2x)^2-6\left(x^2-x+1\right)\right]p^\mu p^\nu+x(1-x)p^\mu p^\nu\end{aligned} \tag{A27}$$

Before evaluating the k integral, we notice that the denominator has the structure $x(1-x)$ which is symmetric with $x\to1-x$, i.e., symmetric with the axis $x=1/2$. So only the pieces symmetric with $x\to1-x$ in the numerator will give non-zero results. We select them by rewriting the x polynomial as a polynomial of $(1-2x)$, and throwing away the odd-power terms. This gives

$$\begin{aligned}N^{\mu\nu}\to&-\frac{(d-2)^2}{d}k^2g^{\mu\nu}+p^2g^{\mu\nu}\left(\frac{6-d}{2}+(d-2)x(1-x)\right)+p^\mu p^\nu\left(2(2-d)x(1-x)+\frac{d-6}{2}\right)\\=&-\frac{(d-2)^2}{d}k^2g^{\mu\nu}-(d-2)x(1-x)p^2g^{\mu\nu}\end{aligned} \tag{A28}$$

The only terms thown is the $(1-2x)/2$ in $(1-x)^2$ in the thrid term of Eq. (A27). We have expressed the result in terms of $x(1-x)$. Then the integration of k gives

$$\begin{aligned}i\Pi_{ab}^{(b+c+d)\mu\nu}(p)=&\frac{i}{(4\pi)^{d/2}}\Gamma(2-d/2)g^2\mu^{2\varepsilon}C_A\delta_{ab}\cdot(p^2g^{\mu\nu}-p^\mu p^\nu)\times\\&\times\int_0^1dx\left(\frac{1}{-x(1-x)p^2-i\varepsilon}\right)^{2-d/2}\left[\frac{6-d}{2}+2(d-2)x(1-x)\right]\\=&\frac{ig^2}{(4\pi)^2}C_A\,\delta_{ab}\cdot(p^2g^{\mu\nu}-p^\mu p^\nu)\left(\frac{4\pi\mu^2}{-p^2-i\varepsilon}\right)^{\varepsilon}\Gamma(\varepsilon)\times\\&\times\int_0^1\left(\frac{1}{x(1-x)}\right)^{\varepsilon}\left[\frac{6-d}{2}+2(d-2)x(1-x)\right]\end{aligned} \tag{A29}$$

We see that here $(p^2 g^{\mu\nu} - p^\mu p^\nu)$ structure appears partly after the integration of x since we already used the property of x integral. Integrating out x then gives

$$\begin{aligned} i\Pi_{ab}^{(b+c+d)\mu\nu}(p) &= \frac{i}{(4\pi)^{d/2}}\Gamma(2-d/2)g^2\mu^{2\varepsilon}C_A\delta_{ab}\cdot(p^2g^{\mu\nu}-p^\mu p^\nu)\times \\ &\quad\times\left[\frac{6-d}{2}B(1-\varepsilon,1-\varepsilon)+2(d-2)B(2-\varepsilon,2-\varepsilon)\right] \\ &= \frac{ig^2}{(4\pi)^2}\,C_A\,\delta_{ab}\cdot(p^2g^{\mu\nu}-p^\mu p^\nu)\left(\frac{5}{3}\frac{S_\varepsilon}{\varepsilon}+\frac{5}{3}\ln\frac{\mu^2}{-p^2-i\varepsilon}+\frac{31}{9}+\mathcal{O}(\varepsilon)\right) \end{aligned} \tag{A30}$$

Similar to fermion loop, $p^2 = 0$ gives a collinear divergence, arising from pinch singularity. This pole cancels the UV pole in DR since taking $p^2 = 0$ results in a scaleless integral (see Eq. (A26)) which is 0 in DR. The UV pole is

$$\mathrm{UV}\ \{i\Pi_{ab}^{(b+c+d)\mu\nu}(p)\} = \delta_{ab}\left(p^2g^{\mu\nu}-p^\mu p^\nu\right)\ \frac{ig^2}{(4\pi)^2}\left(\frac{5}{3}\,C_A\,\frac{S_\varepsilon}{\varepsilon}\right) \tag{A31}$$

This pole will be removed by counterterm diagram (e), and the renormalized two-point function is

$$i\Pi_{ab}^{(ren)(b+c+d)\mu\nu}(p) = \frac{ig^2}{(4\pi)^2}\,C_A\,\delta_{ab}\cdot(p^2g^{\mu\nu}-p^\mu p^\nu)\left(\frac{5}{3}\ln\frac{\mu^2}{-p^2-i\varepsilon}+\frac{31}{9}\right) \tag{A32}$$

Combining with Eq. (A16) and Eq. (A17), we have the total UV pole

$$\mathrm{UV}\ \{i\Pi_{ab}^{\mu\nu}(p)\} = \delta_{ab}\left(p^2g^{\mu\nu}-p^\mu p^\nu\right)\ \frac{ig^2}{(4\pi)^2}\left(\frac{5}{3}\,C_A-\frac{4}{3}\,n_fT_F\right)\frac{S_\varepsilon}{\varepsilon} \tag{A33}$$

and the renormalized two point function

$$i\Pi_{ab}^{(ren)\mu\nu}(p) = \frac{ig^2}{(4\pi)^2}\,C_A\,\delta_{ab}\cdot(p^2g^{\mu\nu}-p^\mu p^\nu)\left[\left(\frac{5}{3}C_A-\frac{4}{3}n_fT_F\right)\ln\left(\frac{\mu^2}{-p^2-i\varepsilon}\right)+\frac{31C_A-20n_fT_F}{9}\right] \tag{A34}$$

f. Counterterm diagram. The counterterm diagram (e) gives a contribution

$$i\Pi_{ab}^{(c.t.)\mu\nu}(p) = -i\,(Z_3-1)\,\delta_{ab}(p^2g^{\mu\nu}-p^\mu p^\nu) \tag{A35}$$

This shall cancel the UV poles of (a-d) in Eq. (A33), so we have

$$Z_3^{[1]} = (Z_3-1)^{[1]} = \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}\,C_A-\frac{4}{3}\,n_fT_F\right)\frac{S_\varepsilon}{\varepsilon} \tag{A36}$$

The renormalized two-point function is in fact

$$i\Pi_{ab}^{(ren)\mu\nu}(p) = i\Pi_{ab}^{(abcd)\mu\nu}(p) + i\Pi_{ab}^{(c.t.)\mu\nu}(p) \tag{A37}$$

As $p^2 = 0$, the bare term vanishes, and all we have is the counterterm contribution, so

$$i\Pi_{ab}^{(ren)\mu\nu}(p) = i\Pi_{ab}^{(c.t.)\mu\nu}(p) = -iZ_3^{[1]}\delta_{ab}(p^2 g^{\mu\nu} - p^\mu p^\nu) \quad \text{as } p^2 = 0 \tag{A38}$$

This then gives

$$\Pi_2(0) = -Z_3^{[1]} = -\frac{g^2}{(4\pi)^2}\left(\frac{5}{3}\,C_A - \frac{4}{3}\,n_f T_F\right)\frac{S_\varepsilon}{\varepsilon} \tag{A39}$$

[1] J. Collins, *Foundations of perturbative QCD*, vol. 32 (Cambridge University Press, 2013), ISBN 978-1-107-64525-7, 978-1-107-64525-7, 978-0-521-85533-4, 978-1-139-09782-6.